Heterotic supersymmetric backgrounds with compact holonomy revisited

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Abstract

We simplify the classification of supersymmetric solutions with compact holonomy of the Killing spinor equations of heterotic supergravity using the field equations and the additional assumption that the 3-form flux is closed. We determine all the fractions of supersymmetry that the solutions preserve and find that there is a restriction on the number of supersymmetries which depends on the isometry group of the background. We examine the geometry of spacetime in all cases. We find that the supersymmetric solutions of heterotic supergravity are associated with a large number of geometric structures which include 7-dimensional manifolds with G_2 structure, 6-dimensional complex and almost complex manifolds, and 4-dimensional hyper-Kähler, Kähler and anti-self-dual Weyl manifolds.

Contents

1 Introduction

The Killing spinor equations of heterotic supergravity have been solved in all cases [1, 2] and it has been found that there are 61 type of solutions up to gauge transformations tabulated in table 2 of [3]. The solutions can be separated into two large classes depending on whether the holonomy, $\mathrm{hol}(\hat\nabla)$, of the connection $\hat\nabla$ with torsion the 3-form flux H is compact or non-compact. The holonomy group describes completely the solution of the gravitino Killing spinor equation (KSE). Each class is further subdivided. This is because not all solutions of the gravitino KSE are also solutions of the dilatino one. If L is the number of $\hat\nabla$-parallel spinors, ie the solutions of the gravitino KSE, then typically the number N of Killing spinors, ie solutions of both gravitino and dilatino KSEs, is $N \leq L$. Backgrounds with $N < L$ have been called descendants in [2].

In the non-compact holonomy case, the solutions of the KSEs are characterized by the pair of numbers (L, N). In particular, $L = 1, 2, 3, 4, 5, 6, 8$, and each L is associated to a unique non-compact holonomy group. Moreover N takes all values $N \leq L$ for each L. In addition for every pair (L, N) there is a unique type of spacetime geometry that can occur. Furthermore, it has been shown that the geometry of the (L, N), $N \neq 7$, backgrounds is a special case of that of (N, N) backgrounds. It suffices therefore to consider only those backgrounds for which all parallel spinors are Killing. This is apart from the $(8, 7)$ case which is treated separately, see also table 4. Since there is a concise description of the geometry of (L, L) backgrounds, the understanding of the geometric conditions imposed by supersymmetry is complete for all backgrounds with non-compact holonomy.

In the compact holonomy case, the solutions of the KSEs can again be labeled by the pair (L, N), where $L = 2, 4, 8, 16$ and $N \leq L$. But, unlike the non-compact case, they are not uniquely characterized by the pair (L, N). In particular for a given pair (L, N), there are different types of geometry that can occur. Moreover, there is no straightforward relation between the geometry of (L, N) and (N, N) backgrounds. Of course all conditions on the spacetime geometry that arise from the KSEs are known [2]. However, they are stated in a non-covariant manner as an artifact of the gauging fixing process used in the context of spinorial geometry method [4] to solving the dilatino KSEs. Furthermore, it is known that the spacetime of such supersymmetric backgrounds admits a Lorentzian Lie algebra action generating Killing vector fields which are nowhere vanishing. Nevertheless, the classification of Lorentzian Lie algebras [5, 6] has not been incorporated in the understanding of geometry of supersymmetric backgrounds.

In this paper, we shall re-examine the dilatino Killing spinor equation in a way that it is manifestly covariant. However, this cannot be achieved without some additional assumptions which we shall explain later. Nevertheless the covariant approach to solving the dilatino KSE has some advantages. One is that we illuminate the large degeneracy of types of geometry which occurs for each pair (L, N), and we find some restriction on N for a given L. The other advantage is that the classification of Lorentzian Lie algebras is now naturally incorporated in the classification of supersymmetric backgrounds.

Different sets of assumptions can be used to solve the dilatino KSE. However the most economical assumption is to take $dH = 0$, ie impose the Bianchi identity of the 3-form field strength, and also use the field equations that arise as the conditions for conformal

invariance at 1-loop in the sigma model perturbation, ie the heterotic supergravity field equations[1]. The condition $dH = 0$ is always valid in the heterotic case at the zeroth order of the α' expansion, and to all orders if the gauge connection is embedded in the spin connection. Moreover, it is valid to all orders for the type II common sector backgrounds.

The above assumptions have far reaching consequences. One reason for this is that one can use holonomy reduction arguments. These are based on the observation that the descendant backgrounds, $N < L$, have more $\hat{\nabla}$-parallel tensors than those with $N = L$ [2]. As a result, the holonomy of $\hat{\nabla}$ for the descendant backgrounds reduces. Examining the pattern of holonomy reduction, one can determine some components of H which in turn allow for the solution of the dilatino KSE. Another consequence of the above assumptions is that one can incorporate information about the classification of Lorentzian Lie algebras into the solution of the KSEs in a natural way. To illustrate this, supersymmetric backgrounds with compact holonomy admit several $\hat{\nabla}$-parallel vector fields one of which is time-like constructed from parallel spinor bi-linears. If $dH = 0$, one can show that their commutator is also $\hat{\nabla}$-parallel. Then an argument based on holonomy reduction requires that the vector space spanned by the vector fields constructed from parallel spinor bi-linears closes under Lie brackets. Since it is a Lorentzian Lie algebra, the classification results can be used to determine the isometries of the supersymmetric backgrounds and as a result the geometry of spacetime. We find that the number of Killing spinors N depends on the Lorentzian Lie algebra which acts on the spacetime.

Assuming that the action of the Lorentzian Lie algebra on the spacetime M can be integrated to a free group action, M is a principal bundle $M = P(G, B; \pi)$ with fibre group G which generates the isometries of spacetime and base space B. Moreover, it is equipped with a principal bundle connection, λ, which twists the fibre G over B [1]. Using these data a brief description of the results we find after solving the KSEs is as follows. In the G_2 case ($L = 2$), there are solutions to the KSEs with 1 and 2 supersymmetries, see also table 4. $\mathfrak{Lie}\, G$ as well as some of the properties of the dilaton for these solutions are summarized in table 1. The base space B is a 7-dimensional manifold with a G_2 structure which is compatible with a connection with skew-symmetric torsion. The connection λ is a G_2 instanton with gauge group G. In the $SU(3)$ case ($L = 4$), there are solutions with 1, 2 and 4 supersymmetries. $\mathfrak{Lie}\, G$ as well as some of the geometric properties of the base space are summarized in table 2. B is a 6-dimensional manifold equipped with either a $SU(3)$ or a $U(3)$ structure compatible with a connection with skew-symmetric torsion. It may also be an almost complex or complex manifold depending on G and N. The connection λ is either a $SU(3)$ or $U(3)$ instanton, ie Hermitian-Einstein connection[2], with gauge group G. In the $SU(2)$ case ($L = 8$), there are solutions with 2, 4, 6 and 8 supersymmetries. $\mathfrak{Lie}\, G$ as well as some of the geometric properties of the base space are summarized in table 3. B is a 4-dimensional manifold which admits either a hyper-Kähler structure, or a Kähler structure or a the Weyl tensor is anti-self-dual. The connection λ is either an anti-self-dual instanton, or a $U(2)$ instanton, or the self-dual part satisfies a Hermitian-Einstein type of condition, with gauge group G.

[1] Alternatively, we can assume the consequences of imposing $dH = 0$ and the field equations but allow for H not to be closed.

[2] In the $U(3)$ case, there is a "cosmological constant" contribution along the diagonal $U(1)$ subgroup.

This paper is organized as follows. In section 2, we review some aspects of the KSEs of heterotic supergravity and specify the part of the dilatino KSE that we solve later. In section 3, we solve the KSEs for backgrounds with holonomy G_2. In section 4, we solve the KSEs for backgrounds with holonomy $SU(3)$. In section 5, we solve the KSEs for backgrounds with holonomy $SU(2)$, and in section 6 we give our conclusions. In appendix A, we describe the solution of the dilatino KSE for group manifolds.

2 Gravitino and dilatino KSEs revisited

Before we proceed to investigate each case separately, we shall first explain the general characteristics of all cases and the strategy we have used to reformulate and solve the dilatino KSE. We summarize some aspects of the gravitino KSE. The solution of the gaugino KSE remains unaltered and can be found in [3], see also [7].

2.1 Gravitino KSE

The gravitino KSE of the heterotic supergravity has been solved in [1, 2]. The spacetime of all supersymmetric backgrounds which admit L parallel spinors with compact isotropy group K in $Spin(9,1)$, $\mathrm{hol}(\hat\nabla) \subseteq K$, admits a local frame $e^A = (e^a, e^i)$ such that the spacetime metric g and H can be written as

$$\begin{aligned} ds^2 &= \eta_{ab}\, e^a e^b + \delta_{ij}\, e^i e^j \ , \\ H &= \frac{1}{3!} H_{abc}\, e^a\wedge e^b\wedge e^c + \frac{1}{2} H_{abi}\, e^a\wedge e^b\wedge e^i + \frac{1}{2} H_{aij}\, e^a\wedge e^i\wedge e^j + \tilde H \ , \end{aligned} \tag{2.1}$$

respectively, where

$$\tilde H = \frac{1}{3!} H_{ijk}\, e^i\wedge e^j\wedge e^k \ , \tag{2.2}$$

and $\eta^{ab} = g^{-1}(e^a, e^b)$. The range of the indices a and i depends on the choice of K. Moreover the gravitino KSE implies that the forms

$$e^a \ , \quad \tau \equiv \frac{1}{k!}\tau_{i_1 i_2\dots i_k} e^{i_1}\wedge e^{i_2}\wedge\cdots\wedge e^{i_k} \ , \tag{2.3}$$

are $\hat\nabla$-parallel, ie

$$\hat\nabla e^a = 0 \ , \quad \hat\nabla \tau = 0 \ , \tag{2.4}$$

where τ stands for the fundamental forms of K. Moreover one of the forms e^a is time-like and all are constructed from $\hat\nabla$-parallel spinor bi-linears.

The first condition in (2.4) implies that

$$\mathcal{L}_a g = 0 \ , \quad de^a = \eta^{ab} i_b H \ , \tag{2.5}$$

where the vector fields $e_a^M = g^{MN}\eta_{ab} e^b_N$ are dual to the 1-forms e^a with respect to the spacetime metric g. The second condition in (2.5) determines the $i_a H$ components of H

in terms of the exterior derivative of e^a. Moreover the second condition in (2.4) can be solved to determine the rest of the components of H in (2.1) in terms of the spacetime metric and e^a and τ and their exterior derivatives. In addition, it leads in some cases to set of conditions on the e^a's and τ's which are interpreted as restrictions on the geometry of spacetime. This is the full content of the gravitino Killing spinor equation, see [1, 2] for more details.

For later use note that if $dH = 0$, the Bianchi identity for $\hat{\nabla}$ yields

$$\hat{R}_{A[B,CD]} = -\frac{1}{3}\hat{\nabla}_A H_{BCD} \ . \tag{2.6}$$

2.2 Dilatino KSE

Before we proceed to organize the conditions that arise from the dilatino KSE, first observe that H in (2.1) and all tensors of M can be decomposed further into irreducible representations of K. This is because K, as containing the holonomy group of $\hat{\nabla}$, acts of the typical fibre of TM, the tangent space of spacetime, with some representation. In particular, the typical fibre decomposes as $\mathbb{R}^{9,1} = \mathbb{R}^{9-\ell,1} \oplus \mathbb{R}^{\ell}$, where $\mathbb{R}^{9-\ell,1}$ is spanned by the directions of the parallel vector fields e_a and the orthogonal complement is taken with respect to the spacetime metric. K acts trivially on $\mathbb{R}^{9-\ell,1}$ and with an irreducible representation on $\mathbb{R}^{\ell}$. Moreover since $K \subset Spin(\ell) \subset Spin(9,1)$, its Lie algebra $\mathfrak{k} \subset \mathfrak{spin}(\ell) = \Lambda^2(\mathbb{R}^{\ell})$. As a result, one can decompose $\Lambda^2(\mathbb{R}^{\ell})$ as $\Lambda^2(\mathbb{R}^{\ell}) = \mathfrak{k} \oplus \mathfrak{k}^{\perp}$. In turn, this will lead to a decomposition of the 2-forms on M. Similarly, all tensors on M can be decomposed in representations of K.

A direct inspection of the conditions which arise in the dilatino KSE for the descendants in [2] reveals that they depend on the tensors

$$[e_a, e_b] \ , \quad (\tilde{d}e^a)|_{\mathfrak{k}^{\perp}} \ , \quad \partial_a \Phi \ , \quad S \ , \tag{2.7}$$

where $\tilde{d}$ is the exterior derivative projected along the e^i directions, S is the singlet in the decomposition of $\tilde{H}$ under K and $(\tilde{d}e^a)|_{\mathfrak{k}^{\perp}}$ denotes the projection of the 2-form $\tilde{d}e^a$ on $\mathfrak{k}^{\perp}$.

On the other hand under the assumptions, $dH = 0$ and $\mathrm{hol}(\hat{\nabla}) \subseteq K$, an analysis of the Bianchi identity (2.6) for the $\hat{\nabla}$ connection reveals that the tensors (2.7) are $\hat{\nabla}$-parallel. Now if the forms (2.7) are linearly independent from those of (2.3), then $\mathrm{hol}(\hat{\nabla})$ reduces to a proper subgroup of K. We shall explain in each case later that the pattern of reduction is

$$G_2 \Longrightarrow SU(3) \Longrightarrow SU(2) \Longrightarrow \{1\} \ . \tag{2.8}$$

So if we assume that (2.7) are linearly independent from those (2.3) for a group K, then simply we have to consider the next case on with more parallel spinors. Since the sequence terminates, to examine all cases it suffices to take (2.7) to be linearly dependent on (2.3). As a result, one has

$$\partial_a \Phi = \mathrm{const} \ , \quad [e_a, e_b] = -H_{ab}{}^c e_c \ , \quad (\tilde{d}e^a)|_{\mathfrak{k}^{\perp}} = f^a\,\tau \ , \quad S = \nu\,\tau \ , \tag{2.9}$$

where f^a and ν are constants and the last two equations are schematic. Therefore it is required that the vector space spanned by vector fields constructed as spinor bi-linears closes under Lie brackets and so it is a Lorentzian Lie algebra. Thus, in particular, $[e_a, e_b]_i = 0$ which implies that

$$H_{abi} = 0 \ . \tag{2.10}$$

This leads to an extensive simplification of the conditions that arise from the dilatino KSE. Moreover, the Lorentzian Lie algebras that arise in each case have been classified and they will be given for each case separately.

Using (2.9) and (2.10), a direct inspection of the conditions that arise from the dilatino Killing spinor equation in [2], reveals that it factorizes[3]. One part gives the condition

$$\theta_\tau = 2\tilde{d}\Phi \ , \tag{2.11}$$

where θ_τ is the Lee form of one of the fundamental forms of K. The expressions of all the relevant Lee forms can be found at [1, 2]. This encompasses the contribution that $\tilde{d}\Phi$ and $\tilde{H}$, apart from the singlet S, make in the dilatino KSE.

The other part of the dilatino KSE involves always the tensors (2.7), and can be written as

$$(\Gamma^a \partial_a \Phi - \frac{1}{12} H_{abc} \Gamma^{abc} - \frac{1}{12} S_{ijk} \Gamma^{ijk} - \frac{1}{4} H_{aij} \Gamma^{aij})\epsilon = 0 \ . \tag{2.12}$$

Observe that in the last term only the $\mathfrak{k}^\perp$ component of $\tilde{d}e^a$ contributes because the spinors ϵ are K-invariant and so are annihilated by the $\mathfrak{k}$ part.

The condition (2.11) is known in all cases. So it remains to solve (2.12). The $K = G_2$ case is simple enough to incorporate the data (2.7) in the calculation of [2] without having to examine (2.12). For $K = SU(3)$ and $SU(2)$, we shall use the classification of Lorentzian Lie algebras [5, 6] to specify the structure constants H_{abc} and then we shall proceed to solve (2.12). This is achieved by writing (2.12) as a sum of commuting operators acting on the K-invariant spinors ϵ and by analyzing their eigen-values and eigen-spaces. The analysis for $K = \{1\}$ has been done in [6], see also [2].

We have argued that the vector space spanned by the parallel vector bi-linears is closed under Lie brackets and so can be identified with a Lie algebra $\mathfrak{Lie}(G)$ of a group G. Moreover $\mathcal{L}_a H = 0$ since $i_a H = de^a$ and $dH = 0$. Assuming that the infinitesimal group action generated by the vector field e_a can be integrated into a free group action on the spacetime[4], the spacetime is a principal bundle $M = P(G, B; \pi)$ with base space B such that

$$\begin{aligned} ds^2 &= \eta_{ab} \lambda^a \lambda^b + \pi^* d\tilde{s}^2 \\ H &= \frac{1}{6} H_{abc} \lambda^a \wedge \lambda^b \wedge \lambda^c + \eta_{ab} \lambda^a \wedge \mathcal{F}^b + \pi^* \tilde{H} \ , \end{aligned} \tag{2.13}$$

[3]This factorization is closely related to the decomposition of the dilatino KSE into the irreducible representation of K on $\mathbb{R}^\ell$ and the singlets.

[4]e_a's never vanish since they are parallel. So the assumption involves the appropriate closure of the orbits generated by the vector fields.

where $\lambda^a \equiv e^a$ is a principal bundle connection and

$$\mathcal{F}^a \equiv \tilde{d}e^a := d\lambda^a - \frac{1}{2}H^a{}_{bc}\lambda^b \wedge \lambda^c \equiv \frac{1}{2}H^a{}_{ij}e^i \wedge e^j \ , \tag{2.14}$$

is the associated curvature. H is the sum of the Chern-Simons form of λ and a 3-form $\tilde{H}$ of B.

Therefore to specify the geometry of the spacetime in each case, one has to describe three pieces of data, (i) the Lorentzian Lie algebra $\mathfrak{Lie}(G)$, (ii) the geometry of the base space B with respect to the pair $(d\tilde{s}^2, \tilde{H})$, and (iii) how the fibre G twists over the base space. The latter is determined by the conditions that the curvature $\mathcal{F}$ of the principal bundle connection satisfies. For the field equations of the theory and details about the notation see [1, 2].

3 G_2

3.1 Holonomy reduction

The backgrounds with $\mathrm{hol}(\hat{\nabla}) \subseteq G_2$ admit three 1-forms e^a that can be constructed from Killing spinor bilinears. Therefore the typical fibre of TM decomposes as $\mathbb{R}^{2,1} \oplus \mathbb{R}^7$. Moreover G_2 acts on $\mathbb{R}^7$ with the 7-dimensional irreducible representation. The isotropy group in G_2 of a vector in $\mathbb{R}^7$ is $SU(3)$. Using the Bianchi identity (2.6) and $dH = 0$, one can show that $[e_a, e_b]$ are $\hat{\nabla}$-parallel [2]. Thus if $[e_a, e_b]$ are linearly independent from $\{e_a\}$, then the holonomy of $\hat{\nabla}$ reduces to a subgroup of $SU(3)$. This is a special case of backgrounds with $SU(3)$ holonomy that will be investigated later. Thus to examine a G_2 case that does not reduce to an $SU(3)$ one, we shall take that the vector space spanned by the e_a's closes under Lie brackets and becomes 3-dimensional Lorentzian Lie algebra. These have been classified and are isomorphic to

$$\mathbb{R}^{2,1} \ , \quad \mathfrak{sl}(2, \mathbb{R}) \ . \tag{3.1}$$

Next $\Lambda^2(\mathbb{R}^7) = \mathfrak{g}_2 \oplus \mathbb{R}^7$. Thus $(de^a)|_{\mathfrak{g}_2^\perp}$ are also in the 7-dimensional representation of G_2. Again the Bianchi identity (2.6) and $dH = 0$ imply that $(de^a)|_{\mathfrak{g}_2^\perp}$ is $\hat{\nabla}$-parallel. Thus if it does not vanish, then the holonomy group again reduces to $SU(3)$. Thus to investigate the G_2 backgrounds which do not reduce to those of $SU(3)$, we must take

$$(de^a)|_{\mathfrak{g}_2^\perp} = 0 \ . \tag{3.2}$$

Now there is only one singlet representation of G_2 in $\Lambda^3(\mathbb{R}^7)$ and this is proportional to the fundamental G_2 3-form φ. Thus we write

$$S_{ijk} = \nu\, \varphi_{ijk} \ . \tag{3.3}$$

Moreover ν is not arbitrary but rather

$$\nu = -\frac{1}{6}(\tilde{d}\varphi, \star\varphi) \ , \tag{3.4}$$

since all $\tilde{H}$ is expressed in terms of the fundamental forms as a consequence of the gravitino KSE[5]. Thus for the G_2 case, equation (2.9) is written as

$$\partial_a \Phi = \text{const} \ , \quad [e_a, e_b] = -H_{ab}{}^c e_c \ , \quad (\tilde{d} e^a)|_{\mathfrak{g}_2^\perp} = 0 \ , \quad S_{ijk} = \nu \, \varphi_{ijk} \ , \tag{3.5}$$

where the structure constants H_{abc} either vanish or are those of $\mathfrak{sl}(2, \mathbb{R})$.

3.2 Dilatino KSE

The (2.11) part of the dilatino KSE is

$$\theta_\varphi = 2 \tilde{d} \Phi \ . \tag{3.6}$$

Using (3.5), (2.12) becomes

$$(\Gamma^a \partial_a \Phi - \frac{1}{12} H_{abc} \Gamma^{abc} - \frac{1}{12} \nu \varphi_{ijk} \Gamma^{ijk}) \epsilon = 0 \ , \tag{3.7}$$

where H_{abc} are the structure constants of either $\mathbb{R}^{2,1}$ or $\mathfrak{sl}(2, \mathbb{R})$. To proceed, we shall examine this equation for $\mathbb{R}^{2,1}$ and $\mathfrak{sl}(2, \mathbb{R})$ separately.

3.2.1 $\mathbb{R}^{2,1}$

In the $\mathbb{R}^{2,1}$ case $H_{abc} = 0$. First suppose that $\nu = 0$. In such case, (3.7) has a solution iff

$$\eta^{ab} \partial_a \Phi \partial_b \Phi = 0 \ . \tag{3.8}$$

Thus there are backgrounds preserving one supersymmetry, $N = 1$, provided that $\partial_a \Phi$ spans a non-vanishing null direction in the Lie algebra $\mathbb{R}^{2,1}$. Therefore the dilaton is linear along a light-cone fibre coordinate z^+ but in general depends non-linearly on the coordinates x of the base space B, ie

$$\Phi = c z^+ + b(x) \ . \tag{3.9}$$

However if $\partial_a \Phi = 0$, then the dilatino KSE vanishes identically and such backgrounds preserve two supersymmetries. The dilaton is constant along the fibre directions but again depends non-linearly on the coordinates of the base space.

Next suppose that $\nu \neq 0$. A direct inspection of the results of [2] reveals that there is an $N = 1$ solution in this case provided that the dilaton is linear along a space-like fibre direction. Take $\partial_1 \Phi \neq 0$, then

$$\partial_1 \Phi = \frac{7}{2} \nu = -\frac{7}{12} (\tilde{d} \varphi, \star \varphi) \tag{3.10}$$

If the dilaton is constant along the fibre directions, ie $\partial_a \Phi = 0$, then $\nu = 0$ and the backgrounds preserve 2 supersymmetries.

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[5] In particular, $\tilde{H} = -\frac{1}{6} (\tilde{d} \varphi, \star \varphi) \varphi + \star \tilde{d} \varphi - \star (\theta_\varphi \wedge \varphi)$, see [8].

3.2.2 $\mathfrak{sl}(2,\mathbb{R})$

If $\mathfrak{Lie}\, G = \mathfrak{sl}(2,\mathbb{R})$, the field equation of the 3-form flux implies that

$$\partial_a \Phi H^a{}_{bc} = 0 \ . \tag{3.11}$$

Thus the $\partial_a\Phi$ direction in $\mathfrak{sl}(2,\mathbb{R})$ commutes with all the others. Since $\mathfrak{sl}(2,\mathbb{R})$ is simple, there is no such direction and so it is required that

$$\partial_a \Phi = 0 \ . \tag{3.12}$$

Thus all backgrounds with $SL(2,\mathbb{R})$ fibre have constant dilaton along the fibre direction. Of course the dilaton still depends on the coordinates on B. A direct inspection of the results of [2] reveals that all such backgrounds preserve 2 supersymmetries[6]. Moreover

$$7(\tilde{d}\varphi, \star\varphi) = H_{-+1} \ , \tag{3.13}$$

where H_{-+1} are the structure constants of $\mathfrak{sl}(2,\mathbb{R})$. The results are summarized in table 1.

$G/d\Phi$	spacelike	null	zero
$\mathbb{R}^{2,1}$	1	1	2
$\mathfrak{sl}(2,\mathbb{R})$	$-$	$-$	2

Table 1: The entries $-$ do not occur. The terms spacelike, null and zero are referred to $d\Phi$ along the group fibre directions. The numerical entries are the number of supersymmetries preserved in each case.

3.3 Geometry

As it has already been mentioned, the spacetime is a principal bundle with fibre $\mathbb{R}^{2,1}$ or $SL(2,\mathbb{R})$. Having specified the geometry of the fibre, it remains to find the geometry of the base space B and how the fibres twist over the base space.

First let us begin with the geometry of the base space. Since for all backgrounds $\tilde{d}e^a|_{\mathfrak{g}_2^\perp} = 0$, as required for the holonomy group not to reduce to $SU(3)$, the fundamental forms of G_2 satisfy

$$i_a\varphi = i_a \star \varphi = 0 \ , \quad \mathcal{L}_a\varphi = \mathcal{L}_a \star \varphi = 0 \ , \tag{3.14}$$

where $\star$ is the Hodge operation of the directions spanned by the e^i's. Therefore both φ and $\star\varphi$ descent on the base space B, and so B has a G_2-structure. This structure is compatible with a connection with skew-symmetric torsion $\hat{\tilde{\nabla}}$ given by the data $(d\tilde{s}^2, \tilde{H})$, where $d\tilde{s}^2 = \delta_{ij}e^ie^j$. Such geometries have been investigated in [8, 1, 2].

The dilatino KSE requires the additional condition

$$(\tilde{d}\varphi, \star\varphi) = 0 \ , \tag{3.15}$$

[6] Solutions of the heterotic KSEs containing $SL(2,\mathbb{R})$ have been emphasized in [9].

for the cases $\mathbb{R}^{2,1}$ $N=2$, and $\mathbb{R}^{2,1}$ $N=1$ when the dilaton is null. In the rest of the cases $(\tilde{d}\varphi, \star\varphi) \neq 0$ and it is related either to the structure constants of $\mathfrak{sl}(2,\mathbb{R})$ or to the spacelike linear dilaton along the group fibres for the $\mathbb{R}^{2,1}$, $N=1$ case.

The fibre twists over the base space with a principal $\mathbb{R}^{2,1}$ or $SL(2,\mathbb{R})$ connection which is a G_2-instanton. This is because $\tilde{d}e^a|_{\mathfrak{g}_2^\perp}=0$ and so

$$\mathcal{F}^a|_{\mathfrak{g}_2^\perp}=0\ . \tag{3.16}$$

Therefore the spacetime can be reconstructed in all cases starting from a 7-dimensional G_2-manifold B compatible with a metric connection with skew-symmetric torsion and a G_2-instanton connection over B with gauge group $\mathbb{R}^{2,1}$ or $SL(2,\mathbb{R})$. It is characteristic that all backgrounds with $N=1$ supersymmetry have a linear dilaton along the fibre group directions.

4 $SU(3)$

4.1 Holonomy Reduction

The backgrounds with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(3)$ admit four 1-forms e^a constructed from parallel spinor bi-linears [1]. So the typical fibre of TM decomposes as $\mathbb{R}^{9,1}=\mathbb{R}^{3,1}\oplus\mathbb{R}^6$ and $SU(3)$ acts on $\mathbb{R}^6\otimes\mathbb{C}=\mathbb{C}^3\oplus\bar{\mathbb{C}}^3$ with the fundamental representation and its complex conjugate. Moreover a vector in $\mathbb{C}^3$ (or $\bar{\mathbb{C}}^3$) has isotropy group $SU(2)$ in $SU(3)$. As in the G_2 case, the Bianchi identity (2.6) and $dH=0$ imply that $[e_a,e_b]$ is $\hat{\nabla}$-parallel. Thus if one of the vector fields $[e_a,e_b]$ is linearly independent from those in $\{e_a\}$, then the holonomy of $\hat{\nabla}$ reduces to a subgroup of $SU(2)$. Such backgrounds are included in those with holonomy $SU(2)$ which will be examined in the next section. To investigate $SU(3)$ backgrounds which do not reduce to $SU(2)$ ones, we shall take that the vector space spanned by $\{e_a\}$ closes under Lie brackets. So $\{e_a\}$'s span a 4-dimensional Lorentzian Lie algebra and these are isomorphic to

$$\mathbb{R}^{3,1}\ ,\quad \mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}\ ,\quad \mathbb{R}\oplus\mathfrak{su}(2)\ ,\quad \mathfrak{cw}_4\ . \tag{4.1}$$

The fundamental forms of $SU(3)$ are the Hermitian 2-form ω and the holomorphic volume 3-form χ which are chosen as

$$\omega=-e^2\wedge e^7-e^3\wedge e^8-e^4\wedge e^9\ ,\quad \chi=(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)\ , \tag{4.2}$$

and so $a,b=0,5,1,6$. To identify the components of $\tilde{d}e^a$ along $\mathfrak{su}(3)^\perp$, we decompose $\tilde{d}e^a$ in (2,0), (1,1) and (0,2)-forms using the almost complex structure constructed from the fundamental form ω of $SU(3)$ and $d\tilde{s}^2$. The directions that lie along $\mathfrak{su}(3)^\perp$ are the (2,0) and (0,2) components as well as the (1,1) component along the Hermitian form ω.

The (0,2) and (2,0) components of $(\tilde{e}^a)|_{\mathfrak{su}(3)^\perp}$ lie in the fundamental representation of $SU(3)$ and its complex conjugate. Again the Bianchi identity (2.6) and $dH=0$ imply that they are $\hat{\nabla}$-parallel. Thus if they do not vanish, then the holonomy of these $SU(3)$ backgrounds reduces to $SU(2)$. So to investigate $SU(3)$ backgrounds which are not special cases of those with holonomy $SU(2)$, we have to set

$$(\tilde{d}e^a)^{2,0}=0\ ; \tag{4.3}$$

the (0,2) component is complex conjugate to (2,0). The component of $(\tilde{d}e^a)|_{\mathfrak{su}(3)^\perp}$ along ω does not reduce the holonomy of the spacetime because it is proportional to a fundamental form of $SU(3)$. Thus we can set

$$(\tilde{d}e^a)|_{\mathfrak{su}(3)^\perp} = f^a \omega \ , \tag{4.4}$$

where f is a constant.

It remains to identify the singlet component S of $\tilde{H}$. In the decomposition of $\Lambda^3(\mathbb{R}^6)\otimes\mathbb{C}$ under $SU(3)$ there is a unique singlet representation proportional to the fundamental form χ and its complex conjugate. Thus we can set

$$S = \frac{1}{2\sqrt{2}}\,(\mu\,\chi + \bar{\mu}\,\bar{\chi}) \ . \tag{4.5}$$

The Bianchi identity (2.6) and $dH = 0$ imply that μ is a complex constant, and the normalization numerical factor has been chosen for convenience.

Therefore, the equation (2.9) for the $SU(3)$ case can be written as

$$\partial_a\Phi = \text{const} \ , \quad [e_a, e_b] = -H_{ab}{}^c e_c \ , \quad (\tilde{d}e^a)|_{\mathfrak{su}(3)^\perp} = f^a\omega \ , \quad S = \frac{1}{2\sqrt{2}}\,(\mu\,\chi + \bar{\mu}\,\bar{\chi}) \ , \tag{4.6}$$

where the structure constants H_{abc} are those of one of the Lie algebras in (4.1).

Before we proceed to examine the dilatino KSE case by case, we shall establish that

$$\partial_a\Phi\, H^a{}_{bc} = 0 \ , \quad f_a\, H^a{}_{bc} = 0 \ . \tag{4.7}$$

The first follows from the field equation of 2-form gauge potential as in the G_2 case. To establish the latter equation, let us compute the Lie derivative of χ along the Killing vector direction e_a and use (4.4) to find

$$\mathcal{L}_a\chi = -3if_a\chi \ . \tag{4.8}$$

Since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, it is easy to see that consistency requires the second equation in (4.7). Therefore the directions spanned by $\partial_a\Phi$ and f are central in $\mathfrak{Lie}\, G$.

4.2 Dilatino KSE

The (2.11) part of the dilatino KSE is

$$\theta_\omega = 2\tilde{d}\Phi \ , \tag{4.9}$$

and the (2.12) can be written as

$$\big(\Gamma^a\partial_a\Phi - \frac{1}{12}H_{abc}\Gamma^{abc} - \frac{1}{4}f_a\omega_{ij}\Gamma^{aij} - \frac{1}{2}(\mu\,\Gamma^{234} + \bar{\mu}\,\Gamma^{\bar{2}\bar{3}\bar{4}})\big)\epsilon = 0 \ , \tag{4.10}$$

where we have expressed the singlet part S of $\tilde{H}$ explicitly in terms of Hermitian gamma matrices.

The simplest case to consider first is $f = \mu = 0$. The dilatino KSE depends only on the dilaton and the structure constants H_{abc}. This is the case examined in [6]. Here the dilatino KSE has been analyzed in detail for the holonomy $SU(2)$ case in appendix A because it is more delicate. The analogous $SU(3)$ case is straightforward. The solutions preserve either 2 or all 4 supersymmetries.

4.2.1 $\mathbb{R}^{3,1}$

$\underline{N=1}$

In this case $H_{abc}=0$. Next suppose that $\partial_a\Phi\neq 0$. Multiplying (4.10) with $\partial_b\Phi\Gamma^b$, we find that it can be written as

$$[(\partial_a\Phi)^2-\frac{1}{2}\partial_a\Phi f^a c(\omega)-\frac{1}{4}\partial_a\Phi f_b c(\omega)\Gamma^{ab}-\frac{1}{2}\partial_a\Phi\Gamma^a c(S)]\epsilon=0\ , \tag{4.11}$$

where

$$c(\omega)=\frac{1}{2}\omega_{ij}\Gamma^{ij}\ ,\quad c(S)=\mu\,\Gamma^{234}+\bar{\mu}\,\Gamma^{\bar{2}\bar{3}\bar{4}}\ . \tag{4.12}$$

Observe that on $SU(3)$-invariant spinors $[c(\omega)]^2=-9\,1_{4\times 4}$ and $[c(S)]^2=-8\mu\bar{\mu}\,1_{4\times 4}$. Then define the traceless matrices

$$A=\frac{1}{2}\partial_a\Phi f^a c(\omega)+\frac{1}{2}\partial_a\Phi\Gamma^a c(S)\ ,\quad B=\frac{1}{4}\partial_a\Phi f_b c(\omega)\Gamma^{ab} \tag{4.13}$$

and observe that on $SU(3)$-invariant spinors

$$A^2=\Delta_1^2\,\mathrm{Id}\ ,\quad B^2=\Delta_2^2\,\mathrm{Id}\ ,\quad AB=BA\ , \tag{4.14}$$

where

$$\begin{aligned}\Delta_1^2&=-\frac{9}{4}(\partial_a\Phi f^a)^2+2\mu\bar{\mu}\partial_a\Phi\partial^a\Phi\ ,\\ \Delta_2^2&=\frac{9}{4}[\partial_a\Phi\partial^a\Phi f_b f^b-(\partial_a\Phi f^a)^2]\ .\end{aligned} \tag{4.15}$$

Now there are various possibilities to consider. First observe that if $\partial_a\Phi\partial^a\Phi\neq 0$, multiplying the (4.10) with $\partial_a\Phi\Gamma^a$ is an invertible operation and so (4.11) is equivalent to (4.10). The null $\partial_a\Phi\partial^a\Phi=0$ and $\partial_a\Phi=0$ cases will be investigated separately, and both of them give backgrounds which preserve at least 2 supersymmetries.

So let us take $\partial_a\Phi\partial^a\Phi\neq 0$. Next if

$$\Delta_1^2,\Delta_2^2>0 \tag{4.16}$$

one can decompose the $SU(3)$-invariant spinors in eigen-spaces of A and B. This in particular implies that $\partial_a\Phi$ and f_a are spacelike and non-co-linear. $N=1$ supersymmetric backgrounds exist provided that

$$(\partial_a\Phi)^2\mp\Delta_1\mp\Delta_2=0 \tag{4.17}$$

the relative ambiguous signs are uncorrelated. For example, $N=1$ solutions exist if both $\partial_a\Phi$ and f_a are orthogonal and space-like.

Next if either

$$\Delta_1^2<0\ ,\quad\text{or}\quad\Delta_2^2<0\ , \tag{4.18}$$

then there are no solutions. This is because the spinors in the heterotic case are real and either A or B can be diagonalized only over complex spinors.

There are two marginal cases to consider depending on whether either Δ_1^2 or Δ_2^2 vanish. In such case either A or B become nilpotent. Notice that f and $\partial_a\Phi$ are either space-like or null. Since the null case will be investigated later, we take $(\partial_a\Phi)^2>0$. To begin, suppose that A is nilpotent, $A^2=0$, ie

$$\Delta_1=0\ ,\quad \Delta_2^2>0\ . \tag{4.19}$$

Decomposing the Killing spinor in eigenvalues of B as $B\epsilon_\pm=\pm\Delta_2\epsilon_\pm$, $\Delta_2>0$, and since A commutes with B, the dilatino KSE becomes

$$\begin{aligned}((\partial_a\Phi)^2-\Delta_2)\epsilon_+-A\epsilon_+ &= 0\\ ((\partial_a\Phi)^2+\Delta_2)\epsilon_--A\epsilon_- &= 0\end{aligned} \tag{4.20}$$

Acting on the second equation with A and using $A^2=0$, we find that $A\epsilon_-=0$. Substituting back, we arrive at $\epsilon_-=0$. Similarly acting on the first equation with A and using a similar argument, we conclude that $\epsilon_+=0$ unless $(\partial_a\Phi)^2-\Delta_2=0$, ie that (4.17) is valid in the limit $\Delta_1=0$ for one choice of sign. Now substituting this into the first equation above, we find that $A\epsilon_+=0$ as well. It remains to find the dimension of the kernel of A on the eigenspace of B with eigenvalue Δ_2. For this first observe that $\partial_a\Phi f^a\neq 0$ since otherwise $\Delta_1=0$ would imply that $\partial_a\Phi$ is null. Using this, the condition $A\epsilon_+=0$ can be rewritten as

$$\big[1-\frac{\partial_a\Phi\Gamma^a c(\omega)c(S)}{9\,\partial_a\Phi f^a}\big]\epsilon_+=0 \tag{4.21}$$

Since the second term is traceless and has eigenvalues ±1 for $\Delta_1=0$, the kernel is 1-dimensional. These solutions are included in (4.17) at the limit that Δ_1 vanishes.

Next take

$$\Delta_1^2>0\ ,\quad \Delta_2=0\ , \tag{4.22}$$

in which case $B^2=0$. As in the previous case, separating the dilatino KSE in eigenspaces, $\epsilon_\pm$, of A with eigenvalues $\pm\Delta_1$, we find that the dilatino KSE is satisfied provided that $\epsilon_-=0$, $B\epsilon_+=0$, and (4.17) is satisfied for $\Delta_2=0$ for one choice of sign. It remains to determine the kernel of B. For this, write $f_a=\ell\partial_a\Phi+u_a$, where $\partial_a\Phi u^a=0$, $u\neq0$, and ℓ is a constant. As in the previous case, we take $(\partial_a\Phi)^2>0$ and so $\Delta_2=0$ implies that u is null. In terms of u B is written as

$$B=\frac{1}{4}\partial_a\Phi u_b c(\omega)\Gamma^{ab}\ . \tag{4.23}$$

Therefore $B\epsilon_+=0$ implies that

$$u_a\Gamma^a\epsilon_+=0 \tag{4.24}$$

and so ϵ satisfies the standard chiral projection condition.

Now suppose that $\partial_a\Phi$ is null. In such case $\Delta_2^2<0$ unless $\partial_a\Phi f^a=0$. Thus in the null case $\Delta_1=\Delta_2=0$. Assume that $f\neq 0$ and act on (4.10) with $f_a\Gamma^a c(\omega)$. This gives

$$[f_a\partial_b\Phi\Gamma^{ab}c(\omega)+\frac{9}{2}f^2-\frac{1}{2}f_a\Gamma^a c(\omega)c(S)]\epsilon=0\ . \tag{4.25}$$

Set $L=\frac{1}{2}f_a\Gamma^a c(\omega)c(S)$ and $K=f_a\partial_b\Phi\Gamma^{ab}c(\omega)$. Now

$$L^2=18f^2\mathrm{Id}\ ,\quad K^2=0\ ,\quad LK=KL\ , \tag{4.26}$$

Thus solutions exist iff $f^2\geq 0$. Moreover acting on (4.25) with K, we find that

$$(\frac{9}{2}f^2-L)K\epsilon=0\ . \tag{4.27}$$

Since the eigenvalues of L are different from $\frac{9}{2}f^2$, the only solutions of this are that either $K\epsilon=0$ or $f^2=0$. In the former case substituting this back into (4.25), one concludes that $\epsilon=0$ and so such backgrounds are not supersymmeric. In the latter case, both f_a and $\partial_a\Phi$ are null and co-linear. Thus acting on (4.10) with $\partial_a\Gamma^a$, one finds that

$$\partial_a\Phi\Gamma^a c(S)\epsilon=0 \tag{4.28}$$

which in turn implies that $\partial_a\Phi\Gamma^a\epsilon=0$, ie without loss of generality one can set

$$\Gamma^+\epsilon=0\ . \tag{4.29}$$

Substituting this back into (4.10), one concludes that there are solutions iff

$$\mu=0 \tag{4.30}$$

ie the (3,0) part of $\tilde{H}$ vanishes. As we shall see this implies that B is a complex manifold. In addition observe that if f and $\partial_a\Phi$ are null and co-linear and $\mu=0$, then the matrices $A=B=0$. As we shall explain such backgrounds preserve at least 2 supersymmetries.

It remains to investigate the case that $\partial_a\Phi=0$. For this, it is easy to see that if $f^2=0$ as well, the dilatino KSE has no solutions unless $\mu=0$. Thus to proceed, we take $f^2\neq 0$. Acting with $f_a\Gamma^a c(\omega)$ on the dilatino KSE and after some rearrangement, it can be rewritten as

$$[1-\frac{f_a\Gamma^a c(\omega)c(S)}{9f^2}]\epsilon=0\ . \tag{4.31}$$

This has solutions provided that

$$f^2=\frac{8}{9}\mu\bar{\mu}\ . \tag{4.32}$$

Moreover such backgrounds preserve at least 2 supersymmetries. Of course if $f=\partial_a\Phi=0$ and $\mu=0$, then the solutions preserve 4 supersymmetries.

<u>$N=2$</u>

We have already seen that if $\partial_a\Phi$ is null or zero, then the solutions preserve 2 supersymmetries. Thus it remains to find the solutions of the Killing spinor equations provided that $(\partial_a\Phi)^2 \neq 0$ In such case, the solutions of the Killing spinor equations will preserve two supersymmetries iff either A or B vanish identically. The possibility of both vanishing is included in the case that $\partial_a\Phi$ is null.

First for $A = 0$, one has that $\partial_a\Phi f^a = 0$ and $\mu = 0$. There are solutions provided that both $\partial_a\Phi$ and f are spacelike and orthogonal, and (4.17) is satisfied. Next $B = 0$, iff $f_a = \ell\partial_a\Phi$, ie f and $\partial_a\Phi$ are co-linear. Solutions preserving 2 supersymmetries exist provided that $\Delta_2^2 > 0$ and (4.17) is satisfied. For these solutions $\mu \neq 0$. There are no solutions preserving 3 supersymmetries.

4.2.2 $\mathbb{R} \oplus \mathfrak{su}(2)$

Without loss of generality, we can assume that $\mathfrak{su}(2)$ spans the directions $1, 5, 6$. Eqn (4.7) implies that the only non-vanishing component of $\partial_a\Phi$ and f_a is $\partial_0\Phi$ and f_0. Setting

$$\frac{1}{6}H_{abc}\Gamma^{abc} = \nu\Gamma^{516} \tag{4.33}$$

where ν is a constant, and acting with Γ^{516} onto the KSE (2.12), one finds

$$[-\partial_0\Phi\Gamma^{0516} + \frac{\nu}{2} + \frac{f_0}{2}\Gamma^{0516}c(\omega) - \frac{1}{2}\Gamma^{516}c(S)]\epsilon = 0 \ . \tag{4.34}$$

Next define

$$A = \frac{1}{2}f_0\Gamma^{0516}c(\omega) \ , \quad B = -\partial_0\Phi\Gamma^{0516} - \frac{1}{2}\Gamma^{516}c(S) \ , \tag{4.35}$$

and observe that

$$A^2 = \frac{9}{4}f_0^2 \ , \quad B^2 = -(\partial_0\Phi)^2 - 2\mu\bar{\mu} \ , \quad AB = BA \ . \tag{4.36}$$

If $(\partial_0\Phi)^2 + 2\mu\bar{\mu} \neq 0$, the eigenvalues of B are complex and there are no solutions. So, we should take

$$\partial_0\Phi = 0 \ , \quad \mu = 0 \ . \tag{4.37}$$

Thus the dilaton is constant along the fibre directions and the Nijenhuis tensor of the base space B vanishes.

In such case, the dilatino KSE becomes

$$[\frac{\nu}{2} + \frac{1}{2}\Gamma^{0516}c(\omega)]\epsilon = 0 \tag{4.38}$$

So there are solutions provided that

$$\nu = \pm 3f_0 \ . \tag{4.39}$$

In fact, all solutions preserve 4 supersymmetries.

4.2.3 $\mathfrak{sl}(2) \oplus \mathbb{R}$

Without loss of generality, let us assume that $\mathfrak{sl}(2)$ spans the directions $0, 5, 1$. In such case, eqn (4.7) implies that the only non-vanishing component of $\partial_a \Phi$ and f_a is $\partial_6 \Phi$ and f_6. Next write

$$\frac{1}{6} H_{abc} \Gamma^{abc} = \nu \Gamma^{051} \tag{4.40}$$

and multiply (4.10) with Γ^{051} to find

$$[\partial_6 \Phi \Gamma^{0516} - \frac{1}{2}\nu - \frac{1}{2} f_6 \Gamma^{0516} c(\omega) - \frac{1}{2} \Gamma^{051} c(S)]\epsilon = 0 \ . \tag{4.41}$$

Then as in the previous case define

$$A = -\frac{1}{2} f_6 \Gamma^{0516} c(\omega) \ , \quad B = \partial_6 \Phi \Gamma^{0516} - \frac{1}{2} \Gamma^{051} c(S) \ , \tag{4.42}$$

and observe that

$$A^2 = \frac{9}{4} f_6^2 \ , \quad B^2 = -(\partial_6 \Phi)^2 + 2\mu\bar{\mu} \ , \quad AB = BA \ . \tag{4.43}$$

Now there are various cases to be considered. If $-(\partial_6 \Phi)^2 + 2\mu\bar{\mu} < 0$, the eigenvalues of B are complex and there are no solutions. On the other hand if $-(\partial_6 \Phi)^2 + 2\mu\bar{\mu} > 0$, there are solutions preserving 1 supersymmetry provided that

$$-\frac{1}{2}\nu \pm \frac{3}{2} f_6 \pm \sqrt{-(\partial_6 \Phi)^2 + 2\mu\bar{\mu}} = 0 \ , \tag{4.44}$$

where the signs are uncorrelated. Notice that if $f_6 = 0$, the solutions preserve 2 supersymmetries.

It remains to investigate the case $(\partial_6 \Phi)^2 = 2\mu\bar{\mu}$. Separating the dilatino KSE on the eigenspaces of A we have

$$[B - \frac{1}{2}\nu \pm \frac{3}{2} f_6]\epsilon_\pm = 0 \ . \tag{4.45}$$

Acting with B and using $B^2 = 0$, there are solutions provided that

$$\nu = \pm 3 f_6 \ . \tag{4.46}$$

Choosing one of the signs, say the positive sign, the Killing spinor equation becomes

$$[\partial_6 \Phi - \frac{1}{2} \Gamma^6 c(S)]\epsilon_+ = 0 \ . \tag{4.47}$$

Now $(\Gamma^6 c(S))^2 = 8\mu\bar{\mu}$, so there are solutions that preserve 1 supersymmetry provided

$$\partial_6 \Phi \pm \sqrt{2\mu\bar{\mu}} = 0 \ . \tag{4.48}$$

Clearly, the $B^2 = 0$ case is included in that of (4.44) for special values of the parameters.

There are backgrounds with 2 supersymmetries if at least one of the operators A or B vanish identically. A vanishes if $f_6 = 0$ and as we have mentioned the solutions preserve 2 supersymmetries. On the other hand B vanishes if both $\partial_6 \Phi$ and $\mu = 0$. In this case the solutions preserve 4 supersymmetries.

4.2.4 $\mathfrak{cw}_4$

Without loss of generality, let us assume that $\mathfrak{cw}_4$ spans the directions $+, -, 1, 6$. In such case, eqn (4.7) implies that the only non-vanishing component of $\partial_a\Phi$ and f_a is $\partial_+\Phi$ and f_+. Setting

$$\frac{1}{6}H_{abc}\Gamma^{abc} = \nu\Gamma^{+16} \ , \tag{4.49}$$

the dilatino KSE can be written as

$$[\partial_+\Phi\Gamma^+ - \frac{1}{2}\nu\Gamma^{+16} - \frac{1}{2}f_+\Gamma^+ c(\omega) - \frac{1}{2}c(S)]\epsilon = 0 \ . \tag{4.50}$$

If $\mu \neq 0$, acting with Γ^+ on the above equation, one finds that

$$c(S)\Gamma^+\epsilon = 0 \tag{4.51}$$

which in turn gives $\Gamma^+\epsilon = 0$. Substituting this into the KSE, one concludes that $c(S)\epsilon = 0$ and so for supersymmetric solutions $\mu = 0$. Therefore the dilatino KSE becomes

$$\partial_+\Phi\Gamma^+ - \frac{1}{2}\nu\Gamma^{+16} - \frac{1}{2}f_+\Gamma^+ c(\omega)]\epsilon = 0 \ . \tag{4.52}$$

Writing $\epsilon = \epsilon_- + \epsilon_+$ with $\Gamma^+\epsilon_+ = 0$, the solutions preserve at least 2 supersymmetries with Killing spinors ϵ_+. To find whether more supersymmetries are preserved substitute $\epsilon = \epsilon_- + \epsilon_+$ into the KSE and observe that

$$[\partial_+\Phi - \frac{1}{2}\nu\Gamma^{16} - \frac{1}{2}f_+ c(\omega)]\Gamma^+\epsilon_- = 0 \ . \tag{4.53}$$

Acting with Γ^{16} on the above equation and taking eigenspaces with respect to $\Gamma^{16}c(\omega)$ and observing that $(\Gamma^{16})^2 = -\mathbf{1}$, there are solutions provided that

$$\partial_a\Phi = 0 \ , \quad \nu = \pm 3 f_+ \ . \tag{4.54}$$

These are in fact the conditions for backgrounds with 4 supersymmetries.

4.3 Geometry

The geometry of the base space B of the spacetime depends on whether μ and f vanish. If $\mu \neq 0$, then the Nijenhuis tensor of the base space does not vanish and so B is an almost complex manifold. On the other hand, solutions with $\mu = 0$ have as base space a complex manifold.

The $SU(3)$ structure of the spacetime is *not* always inherited by the base space B. If $f = 0$, then both ω and χ are invariant under the infinitesimal transformations generated by the Lie algebras (4.1), $\mathcal{L}_a\omega = \mathcal{L}_a\chi = 0$. Since in addition they both vanish along the fibre directions, they descent to a Hermitian form and a holomorphic (3,0)-form on the base B, respectively. Moreover, these data are compatible with $(\tilde{g}, \tilde{H})$. So B is

a manifold with an $SU(3)$ structure compatible with a metric connection with skew-symmetric torsion $\hat{\tilde{\nabla}}$. Such geometries have been investigated extensively in [10]-[18].

Next suppose that $f \neq 0$. In this case again $\mathcal{L}_a \omega = 0$ and it vanishes along the fibre directions of the spacetime, thus it descents to a Hermitian form on B and so B is an almost complex manifold. (It becomes complex if $\mu = 0$.) In addition ω is compatible with $\hat{\tilde{\nabla}}$, $\hat{\tilde{\nabla}}\omega = 0$. This is not the case with χ. Although χ vanishes along the fibre directions of spacetime, the Lie derivative of χ does not (4.8). As a result χ does not descent as a (3,0)-form on B but rather as a (3,0)-form twisted by a line bundle. Moreover χ is not compatible with the data $(\tilde{g}, \tilde{H})$ of the base space. To see this, write $x^M = (y^\alpha, x^\mu)$, where x^μ are coordinates of the base space B and y^α are coordinates of the fibre of M. Then

$$e^a \equiv \lambda^a = \lambda^a{}_\alpha dy^\alpha + \lambda^a{}_\mu dx^\mu \ , \quad e^i = e^i{}_\mu dx^\mu \ . \tag{4.55}$$

The inverse frame is

$$\lambda_a = \lambda_a^\alpha \partial_\alpha \ , \quad e_i = e_i^\mu \partial_\mu - \lambda_\nu^b e_i^\nu \lambda_b^\alpha \partial_\alpha \ . \tag{4.56}$$

In particular

$$\partial_\mu = e_\mu^i \partial_i + \lambda_\mu^a \partial_a \ . \tag{4.57}$$

Thus

$$\hat{\Omega}_\mu{}^i{}_j = e^k{}_\mu \hat{\Omega}_{k,}{}^i{}_j + \lambda^a{}_\mu \hat{\Omega}_{a,}{}^i{}_j = e^k{}_\mu \hat{\Omega}_{k,}{}^i{}_j + \lambda^a{}_\mu H_{aj}{}^i \ . \tag{4.58}$$

Then $\hat{\nabla}_\mu \chi = 0$ implies that

$$\hat{\tilde{\nabla}}_\mu \chi = 3i \lambda_\mu^a f_a \chi \ . \tag{4.59}$$

Thus B does not inherit the $SU(3)$ structure of the spacetime but rather a $U(3)$ structure if $\lambda_\mu^a f_a \neq 0$.

Furthermore, the twisting of the fibre directions of the spacetime over the base space depend on f. If $f = 0$, $\mathcal{F}$ takes values, as a 2-form, in $\mathfrak{su}(3)$ with gauge group one of the groups in (4.1), ie it is a Donaldson connection. However if $f \neq 0$, then $\mathcal{F}$ takes values in $\mathfrak{u}(3)$ with again gauge group one of the groups in (4.1), ie it is a Hermitian-Einstein connection. Some of the geometric properties of these solutions are tabulated in table 2.

4.3.1 $\mathbb{R}^{3,1}$

$\underline{N = 1}$

We have seen that the parameters of solutions with $N = 1$ supersymmetry are restricted as in (4.17), for $\Delta_1, \Delta_2 \geq 0$ and it is required that both A and B do not vanish.

One of the properties of the geometry of these backgrounds is that the base space B of spacetime is always *almost complex*. So see this first recall that if $\partial_a \Phi$ is null or zero, then the solutions preserve at least 2 supersymmetries. Thus, we always have $(\partial_a \Phi)^2 \neq 0$.

$\mathfrak{Lie}\, G/N$	1	2	3	4
$\mathbb{R}^{3,1}$	$\mathrm{AC}, U(3)$	$\mathrm{(A)C}, (S)U(3)$	$-$	$\mathrm{C}, SU(3)$
$\mathbb{R} \oplus \mathfrak{su}(2)$	$-$	$-$	$-$	$\mathrm{C}, U(3)$
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$	$\mathrm{AC}, U(3)$	$\mathrm{AC}, SU(3)$	$-$	$\mathrm{C}, U(3)$
$\mathfrak{cw}_4$	$-$	$\mathrm{C}, (S)U(3)$	$-$	$\mathrm{C}, U(3)$

Table 2: N is the number of supersymmetries and $\mathfrak{Lie}\, G$ is the Lie algebra of isometries of the solutions. The entries give information about the geometry of base space. AC stands for almost complex and C for complex manifold, respectively. (A)C stands for either complex or almost complex. The groups $SU(3)$ and $U(3)$ denote the holonomy of $\hat{\tilde{\nabla}}$. $(S)U(3)$ means that the holonomy of $\hat{\tilde{\nabla}}$ is either contained in $SU(3)$ or $U(3)$. The entries $-$ do not occur.

Now if $\mu = 0$, Δ_1^2 is negative unless $\partial_a \Phi f^a = 0$. Thus for solutions to exist, in addition $\partial_a \Phi f^a = 0$. In such case $A = 0$ and the solutions preserve at least 2 supersymmetries.

Another consequence of the analysis above is that the dilaton is linear for all $N = 1$ backgrounds along the fibre directions. This follows from the requirement that $(\partial_a \Phi)^2 \neq 0$.

It is also straightforward to observe that if $f = 0$, then $B = 0$ and so again the solutions preserve at least 2 supersymmetries. *Thus we have shown that for solutions with* $N = 1$ *supersymmetry,* B *is an almost complex manifold with* $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq U(3)$.

<u>$N = 2$</u>

Solutions with 2 supersymmetries may or may not admit a complex base space B. There are two classes of solutions with complex base space ($\mu = 0$) the following:

- f and $\partial_a \Phi$ null and co-linear, $f_a = \ell \partial_a \Phi$, $\ell \geq 0$, $\partial_a \Phi \neq 0$.
- f and $\partial_a \Phi$ spacelike and orthogonal, $\partial_a \Phi f^a = 0$, $\partial_a \Phi \neq 0$.

There are also solutions with 2 supersymmetries and almost complex base space. Again, there are two classes

- $\partial_a \Phi$ spacelike and $f_a = \ell \partial_a \Phi$, $\ell \geq 0$.
- $\partial_a \Phi = 0$, $f \neq 0$.

Of course in all cases, apart from the last one, the parameters are required to satisfy (4.17).

Therefore the base space of backgrounds with 2 supersymmetries may or may not be a complex manifold and may or may not have $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(3)$. So there is a large range of possibilities. Moreover, there are backgrounds for which the dilaton is constant along the fibre directions but for these B is almost complex and $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq U(3)$.

There are no backgrounds with 3 supersymmetries. For backgrounds with 4 supersymmetries, the dilaton is constant along the fibre directions, B is complex, and $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(3)$.

4.3.2 $\mathbb{R} \oplus \mathfrak{su}(2)$

All solutions in this case preserve 4 supersymmetries. It follows from [1] that the base space B is complex[7] but $\mathrm{hol}(\hat{\hat{\nabla}}) \subseteq U(3)$.

4.3.3 $\mathfrak{sl}(2) \oplus \mathbb{R}$

$\underline{N=1}$

We have seen that the parameters of solutions with $N=1$ supersymmetry are restricted as in (4.44), for $2\mu\bar{\mu} \geq (\partial_6\Phi)^2$ and it is required that both A and B do not vanish.

It is clear from the conditions of the dilatino KSE that for all $N=1$ backgrounds the base space is *an almost complex manifold and* $\mathrm{hol}(\hat{\hat{\nabla}}) \subseteq U(3)$. However unlike the $\mathbb{R}^{3,1}$ case, there are backgrounds with 1 supersymmetry and constant dilaton in the fibre directions, $\partial_a\Phi = 0$.

$\underline{N=2}$

The base space B of solutions with 2 supersymmetries is always an almost complex manifold. However in this case $f=0$. As a result the base space B is *an almost complex manifold with* $\mathrm{hol}(\hat{\hat{\nabla}}) \subseteq SU(3)$. There are also solutions with constant dilaton along the fibre directions $\partial_a\Phi = 0$.

For solutions with 4 supersymmetries B is a complex manifold, $f \neq 0$. The geometry has been investigated in [1].

4.3.4 $\mathfrak{cw}_4$

There are backgrounds with 2 or 4 supersymmetries. In particular, there are no $N=1$ solutions. For all $N=2$ solutions the base space is complex $\mu=0$. Moreover, there are solutions with constant or non-constant dilaton and with either $f=0$ or $f \neq 0$, ie either $\mathrm{hol}(\hat{\hat{\nabla}}) \subseteq SU(3)$ or $\mathrm{hol}(\hat{\hat{\nabla}}) \subseteq U(3)$.

The base space of solutions with $N=4$ supersymetries is a complex manifold and $\mathrm{hol}(\hat{\hat{\nabla}}) \subseteq U(3)$. This is because f does not vanish as it can be seen in (4.54).

In all cases $SU(3)$ we have investigated above, there are no solutions with 3 supersymmetries. This is reminiscent to the absence of solutions preserving 31 supersymmetries in IIB, IIA and 11-dimensional supergravities [19, 20]. However we have not ruled out the possibility that solutions with 3 supersymmetries exist after an appropriate discrete identification of solutions that preserve 4 supersymmetries as in [21].

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[7] The conditions for solutions with 4 supersymmetries are that B is complex and $\partial_a\Phi = 0$, $\theta_\omega = 2\tilde{d}\Phi$, $H_{aij} = H_{akl}I^k{}_iI^l{}_j$ and $\frac{1}{3}\epsilon_a{}^{bcd}H_{bcd} - \omega^{ij}H_{aij} = 0$.

5 $SU(2)$

5.1 Holonomy reduction

The backgrounds with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(2)$ admit six 1-forms e^a constructed from parallel spinor bi-linears. So the typical fibre of TM decomposes as $\mathbb{R}^{9,1} = \mathbb{R}^{5,1} \oplus \mathbb{R}^4$ and $SU(2)$ acts on $\mathbb{R}^4 \otimes \mathbb{C} = \mathbb{C}^2 \oplus \bar{\mathbb{C}}^2$ with the fundamental representation and its complex conjugate. Moreover a vector in $\mathbb{C}^2$ (or $\bar{\mathbb{C}}^2$) has isotropy group $\{1\}$ in $SU(2)$. As in the previous cases, the Bianchi identity (2.6) and $dH = 0$ imply that $[e_a, a_b]$ are $\hat{\nabla}$-parallel. Thus if one of the vector field $[e_a, e_b]$ is linearly independent from those in $\{e_a\}$, then the holonomy of $\hat{\nabla}$ reduces to $\{1\}$ and it becomes a special case of backgrounds with holonomy $\{1\}$. These have been classified in [6], see also [2]. Thus to investigate $SU(2)$ backgrounds which do not reduce to $\{1\}$ ones, we shall take that the vector space spanned by $\{e_a\}$ to close under Lie brackets. So $\{e_a\}$'s span a 6-dimensional Lorentzian Lie algebra and these are isomorphic to

$$\mathbb{R}^{5,1} \ , \quad \mathbb{R}^{3,1} \oplus \mathfrak{su}(2) \ , \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^4 \ , \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \ , \quad \mathfrak{cw}_4 \oplus \mathbb{R}^2 \ , \quad \mathfrak{cw}_6 \ . \tag{5.1}$$

The fundamental forms of $SU(2)$ are the three Hermitian forms

$$\omega_1 = -e^3 \wedge e^8 - e^4 \wedge e^9 \ , \quad \omega_2 = e^3 \wedge e^4 - e^8 \wedge e^9 \ , \quad \omega_3 = -e^4 \wedge e^8 + e^3 \wedge e^9 \tag{5.2}$$

therefore $a = 0, 5, 1, 6, 2, 7$. These are associated with endomorphism which satisfy the algebra of imaginary unit quaternions, $I_r I_s = -\delta_{rs} 1_{4\times4} + \epsilon_{rst} I_t$.

To identify the components of $\tilde{d}e^a$ along $\mathfrak{su}(2)^\perp$, we use the decomposition $\Lambda^2(\mathbb{R}^4) = \mathfrak{su}(2) \oplus \mathfrak{su}^\perp(2)$, where $\mathfrak{su}^\perp(2) = \mathfrak{su}(2)$. The $\mathfrak{su}(2)$ component is spanned by the anti-self-dual 2-forms while the $\mathfrak{su}^\perp(2)$ is spanned the self-dual 2-forms given in (5.2). Therefore, the $\mathfrak{su}^\perp(2)$ component of $\tilde{d}e^a$ can be written as

$$\tilde{d}e^a + \star\tilde{d}e^a = 2 f^a_r \omega^r \ . \tag{5.3}$$

Since the Bianchi identity (2.6) and $dH = 0$ imply that $\tilde{d}e^a + \star\tilde{d}e^a$ are $\hat{\nabla}$-parallel, f^a_r are some real constants.

It can be easily seen that there is no a $\mathfrak{su}(2)$-invariant component of $\tilde{H}$, and so $S = 0$. Thus for the $SU(2)$ case, equation (2.9) is written as

$$\partial_a \Phi = \mathrm{const} \ , \quad [e_a, e_b] = -H_{ab}{}^c e_c \ , \quad \tilde{d}e^a + \star\tilde{d}e^a = 2 f^a_r \omega^r \ , \quad S = 0. \tag{5.4}$$

As in the $SU(3)$ case, we have that $\partial_a \Phi$ and f are restricted. In particular one finds that

$$\partial_a \Phi H^a{}_{bc} = 0 \ , \tag{5.5}$$

and

$$- H^c{}_{ab} f_{cs} = 2 f_{ar} f_{bt} \epsilon^{rt}{}_s \ . \tag{5.6}$$

The former equation follows from the field equation of the 2-form gauge potential as in the G_2 and $SU(3)$ cases. To prove (5.6) use $\hat{\nabla}\omega^r = 0$, the quaternionic algebra of I_r's and $\tilde{de}^a + \star\tilde{de}^a = 2f^a_r\omega^r$ in (5.4) to find that

$$\mathcal{L}_a\omega^r = 2f_{as}\epsilon^s{}_{rt}\,\omega^t\ . \tag{5.7}$$

Moreover the property $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ of the Lie derivative implies (5.6). Therefore $f:\ \mathfrak{Lie}\,G \to \mathfrak{su}(2)$ is a Lie algebra homomorphism.

5.1.1 Solution of the homomorphism condition (5.6)

Depending on $\mathfrak{Lie}\,G$ in (5.1), (5.6) has two non-vanishing solutions. One solution is

$$f_{ar} = w_a v_r\ ,\quad v_r v^r = 1\ , \tag{5.8}$$

provided that

$$H^c{}_{ab}\,w_c = 0\ . \tag{5.9}$$

Clearly the direction along w in $\mathfrak{Lie}\,G$ is central. So this solution exists for all $\mathfrak{Lie}\,G$ in (5.1) apart from $\mathfrak{sl}(2)\oplus\mathfrak{su}(2)$.

The other solution is

$$(f_{as}) = (0,0,0,-\frac{1}{2}H_{627}\tilde{f}_{r's})\ ,\quad r' = 6,2,7\ ,\quad s = 1,2,3\ , \tag{5.10}$$

provided that $\mathfrak{Lie}\,G = \mathfrak{h}\oplus\mathfrak{su}(2)$, $\mathfrak{su}(2)$ spans the directions $6,2,7$ and $(\tilde{f}_{r's}) = \mathrm{diag}(1,1,1)$. Therefore f is a Lie algebra homomorphism with Kernel $\mathfrak{h}$. Thus when it is restricted on the $\mathfrak{su}(2)$ subalgebra of $\mathfrak{Lie}\,G$, it is a Lie algebra isomorphism.

To prove (5.8) and (5.10), first observe that if there is a $t\in\mathfrak{Lie}\,G$ which is central and $t^a f_{ar}\neq 0$, then all the other f_{ar} are proportional to $t^a f_{ar}$, ie $f_{br} = u_b\,t^a f_{ar}$. This follows easily by contracting (5.6) with t. In particular one has $|v|v_r = t^a f_{ar}$ in (5.8).

Moreover if $\mathfrak{Lie}\,G$ contains a $\mathfrak{cw}$ algebra, all solutions of (5.6) are of the type (5.8). To show this, observe that the non-vanishing structure constants of $\mathfrak{cw}$ are of the type H_{+ij}. If (f_{-r}) does not vanish, then it follows from the previous statement that the solution is (5.8). Next suppose that $f_{-r} = 0$. Setting $a = i$ and $b = j$ in (5.6) and using the only H_{+ij} are non-vanishing, one finds that $f_i = (f_{ir})$ are proportional to each other, ie $f_{ir} = u_i v_r$. Then setting $a = +$ and $b = j$ in (5.6) and using the proportionality of f_i's, it is easy to see that $f_+ = (f_{+r})$ is also proportional to f_i's. Thus the only solution is (5.8).

It remains to investigate (5.6) for the Lie algebras $\mathfrak{Lie}\,G = \mathfrak{h}\oplus\mathfrak{su}(2)$ in (5.1). Now if $t^a f_{ar} = 0$ for all t central elements, then one derives (5.10) for the $\mathbb{R}^{2,1}\oplus\mathfrak{su}(2)$ case. Next consider $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)$. Using the fact that $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ commute, if $f_a = (f_{ar})$ has non-vanishing components for directions in both these subalgebras, then one concludes the only solution is (5.8). But since w is required in addition to be central, this solution is excluded. Alternatively $f_a = (f_{ar})$ must vanish when restricted on either $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$. In addition the Kernel must be $\mathfrak{sl}(2,\mathbb{R})$ since f is a Lie algebra homomorphism. Thus $(f_{as}) = (0, f_{rs})$, where (f_{rs}) is an invertible matrix.

To proceed, we can always choose e^a such that η_{ab} is the Minkowski metric. Without loss of generality, we orient $\mathfrak{su}(2)$ in the directions $6, 2$ and 7, and so the structure constants are $H_{r's't'} = H_{627}\epsilon_{r's't'}$, $r', s', t' = 6, 2, 7$, $\epsilon_{627} = 1$. Next setting

$$f_{r'r} = -\frac{1}{2} H_{627} \tilde{f}_{r'r} \ , \tag{5.11}$$

one finds that

$$\tilde{f}_{r'r} \tilde{f}_{s's} \epsilon^{rs}{}_t = \epsilon_{r's'}{}^{t'} \tilde{f}_{t't} \ . \tag{5.12}$$

It is easy to see that the 3-vectors $\tilde{f}_{r'} = (\tilde{f}_{r'r})$ have unit length and are mutually orthogonal. Therefore up to an orthogonal transformation, one can set $\tilde{f}_{61} = \tilde{f}_{22} = \tilde{f}_{73} = 1$ and the rest of the components to vanish. This proves (5.10).

5.1.2 Dilatino KSE

Adapting the general analysis in section 2 to this case, the part of the dilatino KSE involving the Lee form (2.11) is

$$\theta_{\omega_1} = \theta_{\omega_2} = \theta_{\omega_3} = 2\tilde{d}\Phi \ . \tag{5.13}$$

Using (5.4), the rest of the KSE (2.12) can be written as

$$(\Gamma^a \partial_a \Phi - \frac{1}{12} H_{abc} \Gamma^{abc} - \frac{1}{4} f_{ar}\, \omega^r_{ij}\, \Gamma^{aij})\epsilon = 0 \ . \tag{5.14}$$

In what follows we shall solve (5.14) for the Lie algebras (5.1), and for the solutions of the homomorphism condition (5.8) and (5.10).

5.2 $f = wv$

The dilatino KSE in the (5.8) becomes

$$(\Gamma^a \partial_a \Phi - \frac{1}{12} H_{abc} \Gamma^{abc} - \frac{1}{4} w_a v_r (\omega^r)_{ij} \Gamma^{aij})\epsilon = 0 \ . \tag{5.15}$$

The solutions depend on the properties of w and $\partial\Phi$, and so we shall examine various cases. We shall always assume that $w \neq 0$ since the $w = 0$ case is examined in appendix A. We shall demonstrate that there are solutions that preserve either 4 or 8 supersymmetries.

5.2.1 $\mathbb{R}^{5,1}$

$w^2 \neq 0$

Since $\mathbb{R}^{5,1}$ is abelian $H_{abc} = 0$. Act on (5.15) with $w_a \Gamma^a$ and write the resulting equation as

$$(\partial_a \Phi w^a + A - w^2 B)\epsilon = 0 \tag{5.16}$$

where

$$A = w_a \partial_b \Phi \Gamma^{ab} \ , \quad B = \frac{1}{4} v_r (\omega^r)_{ij} \Gamma^{ij} \ . \tag{5.17}$$

Observe that

$$A^2 = -\big[w^2 (\partial_a \Phi)^2 - (w^a \partial_a \Phi)^2\big] 1_{8\times 8} \ , \quad B^2 = -1_{8\times 8} \ , \quad AB = BA \ . \tag{5.18}$$

Since B has always imaginary eigenvalues act with B on (5.16) to find

$$(\partial_a \Phi w^a B + AB + w^2)\epsilon = 0 \ . \tag{5.19}$$

Separating the equation in eigenvalues of AB, there are solutions iff

$$\partial_a \Phi \, w^a = 0 \ , \quad w^2 (\partial_a \Phi)^2 > 0 \ , \tag{5.20}$$

and

$$\pm \sqrt{w^2 (\partial_a \Phi)^2} = w^2 \ . \tag{5.21}$$

The first condition is required so that the dependence on B to vanish because otherwise the eigen-spaces are complex. The second condition is required for the eigenvalues of AB to be real. The last condition is necessary for matching the eigenvalues of AB with w^2. All these conditions have solutions, iff both w and $\partial_a \Phi$ are spacelike[8] and orthogonal. So $w^2 = (\partial_a \Phi)^2$. Note that if $\partial_a \Phi$ vanishes, for consistency w is null and this case is investigated below. The solutions preserve 4 supersymmetries.

$\underline{w^2 = 0}$

Acting on (5.14) with $\Gamma^a w_a$ as before, one finds that

$$(\partial \Phi \cdot w + A)\epsilon = 0 \tag{5.22}$$

Acting now on (5.14) with $\Gamma^a \partial_a \Phi$ and using the above equation, one concludes that

$$((\partial_a \Phi)^2 - 2 \partial \Phi \cdot w B)\epsilon = 0 \tag{5.23}$$

This has no solutions unless $(\partial_a \Phi)^2 = \partial \Phi \cdot w = 0$. Thus both $\partial_a \Phi$ and w must be null and co-linear.

The analysis for the case that $(\partial_a \Phi)^2 = 0$ is similar to that for which $w^2 = 0$ leading to the same solution that $\partial_a \Phi$ and w must be null and co-linear. The solutions preserve 4 supersymmetries. The Killing spinors satisfy the projection $w_a \Gamma^a \epsilon = 0$.

5.2.2 $\mathbb{R}^{2,1} \oplus \mathfrak{su}(2)$

$\underline{w^2 \neq 0}$

[8] The case that both are timelike is excluded by the orthogonality condition.

Define A and B as before and moreover set

$$C = w_a \partial_b \Phi \Gamma^{ab} - \frac{1}{12} w_{a_1} H_{a_2 a_3 a_4} \Gamma^{a_1 a_2 a_3 a_4} \ . \tag{5.24}$$

Observe that the two terms in C anti-commute with each other and

$$C^2 = \Delta^2 \, 1_{8\times 8} \tag{5.25}$$

where

$$\Delta^2 = -\big[w^2 (\partial_a \Phi)^2 - (w^a \partial_a \Phi)^2\big] + \frac{1}{4} w^2 H^2 \ , \quad H^2 = \frac{1}{6} H_{abc} H^{abc} > 0 \ . \tag{5.26}$$

Moreover we have that

$$BC = CB \ . \tag{5.27}$$

Acting on (5.14) with $w_a \Gamma^a$, observe that it can be rewritten as

$$(w^a \partial_a \Phi + C - w^2 B)\epsilon = 0 \ . \tag{5.28}$$

Since the eigenvalues of B are imaginary, acting with B on (5.28) and separating the equation in eigenspaces of BC, there are solutions iff

$$\Delta^2 < 0 \ , \quad w^a \partial_a \Phi = 0 \ , \quad \sqrt{-\Delta^2} \pm w^2 = 0 \ . \tag{5.29}$$

The first condition is required for BC to have real eigenvalues. Since $\partial_a \Phi$ is orthogonal to w, Δ^2 simplifies as

$$\Delta^2 = -w^2 (\partial_a \Phi)^2 + \frac{1}{4} w^2 H^2 \ . \tag{5.30}$$

Now suppose that $w^2 > 0$. In such case $\Delta^2 < 0$ requires that $(\partial_a \Phi)^2 > \frac{1}{4} H^2 > 0$. Thus there are solutions that preserve 4 supersymmetries provided that w and $\partial_a \Phi$ are both spacelike and orthogonal and

$$(\partial_a \Phi)^2 - \frac{1}{4} H^2 = w^2 \ . \tag{5.31}$$

Next if $w^2 < 0$, the condition $\Delta^2 < 0$ requires that $(\partial_a \Phi)^2 < \frac{1}{4} H^2$. Again there are solutions preserving 4 supersymmetries provided that $\partial_a \Phi$ is spacelike. There are no solutions with both w and $\partial_a \Phi$ timelike because of the orthogonality condition. Note also that if $\partial_a \Phi$ is null, there is no solution.

$\underline{w^2 = 0}$

Acting with $w_a \Gamma^a$ on (5.14) and using $w^2 = 0$, one finds that

$$[w \cdot \partial \Phi + w_a \partial_b \Phi \Gamma^{ab} - \frac{1}{12} w_a H_{bcd} \Gamma^{abcd}]\epsilon = 0 \tag{5.32}$$

On the other hand, acting with $\partial_a\Phi\Gamma^a - \frac{1}{12}H_{abc}\Gamma^{abc}$ on (5.14) and using the above equation, one finds that

$$[(\partial_a\Phi)^2 - \frac{1}{24}H_{abc}H^{abc} - \frac{1}{2}w\cdot\partial\Phi B]\epsilon = 0 \ . \tag{5.33}$$

This implies that

$$(\partial_a\Phi)^2 = \frac{1}{24}H_{abc}H^{abc} > 0 \ , \quad w\cdot\partial\Phi = 0 \ . \tag{5.34}$$

Since $\partial_a\Phi$ is required to be spacelike, we act on (5.14) with $\partial_a\Phi\Gamma^a$ to find

$$[(\partial_a\Phi)^2 - \frac{1}{12}\partial_a\Phi H_{bcd}\Gamma^{abcd} + w_a\partial_b\Phi\Gamma^{ab}B]\epsilon = 0 \ . \tag{5.35}$$

Next observe that the last two terms in the above equation anti-commute and so define

$$D = \frac{1}{12}\partial_a\Phi H_{bcd}\Gamma^{abcd} - w_a\partial_b\Phi\Gamma^{ab}B \ . \tag{5.36}$$

In terms of D the dilatino KSE (5.14) can be written as

$$[(\partial_a\Phi)^2 - D]\epsilon = 0 \ . \tag{5.37}$$

Observe that

$$D^2 = \frac{1}{4}(\partial_a\Phi)^2H^2 1_{8\times 8} \ . \tag{5.38}$$

Therefore the eigenvalues of D are real and so there are solutions which preserve 4 supersymmetries provided that (5.34) is satisfied. In fact the Killing spinors satisfy the projection $w_a\Gamma^a\epsilon = 0$. The remaining dilatino KSE is also satisfied because the $SU(2)$-invariant spinors are chiral from the 6-dimensional perspective.

5.2.3 $\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}^3$

The analysis of the solutions of the dilatino Killing spinor equation is similar to that of the previous case. So we define again A, B, C and D. The only difference is that we now have to take into account that $\partial_a\Phi$ and w are spacelike and H is timelike.

$\underline{w^2 \neq 0}$

A brief inspection of the previous case reveals that there is a solution of the Killing spinor equations preserving 4 supersymmetries provided that both $\partial_a\Phi$ and w are spacelike and orthogonal, and (5.31) is satisfied. Observe that in this case, there are solutions for which $\partial_a\Phi = 0$, ie the dilaton is constant along the fibre directions.

5.2.4 $\mathfrak{cw}_4 \oplus \mathbb{R}^2$ and $\mathfrak{cw}_6$

Suppose now that $\mathfrak{Lie}G = \mathfrak{cw}_6$ or $\mathfrak{cw}_4 \oplus \mathbb{R}^2$. In this case H is null. Acting with $H_{abc}\Gamma^{abc}$ on (5.14), one finds that

$$\big(- \frac{1}{12} H_{abc}\partial_d\Phi\Gamma^{abcd} + \frac{1}{48} H_{abc} w_d \Gamma^{abcd} v_r \omega^r_{ij}\Gamma^{ij}\big)\epsilon = 0 \ . \tag{5.39}$$

Next acting on the same equation with $\partial_a\Phi\Gamma^a - \frac{1}{4} w_a\Gamma^a v_r\omega^r_{ij}\Gamma^{ij}$ and using the above equation, one finds that

$$(\partial_a\Phi)^2 = w^2 \ , \quad \partial\Phi\cdot w = 0 \ . \tag{5.40}$$

So there are two possibilities. Either $\partial_a\Phi$ and w are null and co-linear or both are spacelike and orthogonal. In the former case, the backgrounds preserve 4 of supersymmetries. The Killing spinors satisfy the projection $w_a\Gamma^a\epsilon = 0$. Now if both $w = \partial_a\Phi = 0$ and the structure constants of $\mathfrak{cw}_6$ are self-dual, then the solution preserves 8 supersymmetries.

The latter case is only available for $\mathfrak{Lie}G = \mathfrak{cw}_4 \oplus \mathbb{R}^2$. To investigate it further act on the dilatino KSE with $w_a\Gamma^a$ and observe that it can be written as $(C - w^2B)\epsilon = 0$ or equivalently as

$$(CB + w^2)\epsilon = 0 \ . \tag{5.41}$$

Moreover $(CB)^2 = w^2(\partial_a\Phi)^2 1_{8\times 8}$. Since both $\partial_a\Phi$ and w are spacelike CB has real eigenvalues and there are solutions preserving 4 supersymmetries provided that (5.40) is satisfied.

5.3 Solutions for f in (5.10)

5.3.1 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$

In the $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$ case, $\partial_a\Phi = 0$. The KSE becomes

$$[-\frac{1}{12} H_{abc}\Gamma^{abc} - \frac{1}{4} f_{ar}\Gamma^a\omega^r_{ij}\Gamma^{ij}]\epsilon = 0 \ . \tag{5.42}$$

Without loss of generality, let us take $\mathfrak{su}(2)$ to be along the directions 051 and $\mathfrak{su}(2)$ along the directions 627. In such case, the Killing spinor equation can be rewritten as

$$\big[- \frac{1}{2}[H_{051} + H_{627}]\Gamma^{627} - \frac{1}{4} f_{r'r}\Gamma^{r'}\omega^r_{ij}\Gamma^{ij}\big]\epsilon = 0 \ . \tag{5.43}$$

Using (5.10) and acting with Γ^{627}, one finds

$$\big[\frac{1}{2}[H_{051} + H_{627}] - \frac{\nu}{8}\epsilon_{r's't'}\Gamma^{s't'}\omega^{r'}_{ij}\Gamma^{ij}\big]\epsilon = 0 \ , \quad \nu = -\frac{1}{2} H_{627} \ , \tag{5.44}$$

where $\epsilon_{627} = 1$ and $\omega^6 \equiv \omega^1$, $\omega^2 \equiv \omega^2$ and $\omega^7 \equiv \omega^3$. Next one can show that on the space of $SU(2)$-invariant spinors

$$W^2 = -2W + 3 \ , \quad W = \frac{1}{8}\epsilon_{r's't'}\Gamma^{s't'}\omega^{r'}_{ij}\Gamma^{ij} \ . \tag{5.45}$$

Therefore W has eigenvalues 1 and -3. Since W is traceless, 1 has multiplicity 6 while -3 has multiplicity 2. Therefore if

$$H_{051} + 2H_{627} = 0 \tag{5.46}$$

the background preserves 6 supersymmetries, while if

$$H_{051} - 2H_{627} = 0 \tag{5.47}$$

the background preserves 2 supersymmetries. In the special case that the structure constants of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$ are self-dual and $f = 0$, the solutions preserve all 8 supersymmetries.

5.3.2 $\mathbb{R}^{2,1} \oplus \mathfrak{su}(2)$

Next let us consider $\mathbb{R}^{2,1} \oplus \mathfrak{su}(2)$. In this case, the Killing spinor equation can be written as

$$[Z + \frac{1}{2}H_{627} - \nu W]\epsilon = 0 \ , \quad \nu = -\frac{1}{2}H_{627} \ , \tag{5.48}$$

where $Z = \Gamma^{627}\partial_a\Phi\Gamma^a$ and W is given in (5.45). Since $\partial_a\Phi$ is in the directions of $\mathbb{R}^{2,1}$, observe that

$$Z^2 = (\partial_a\Phi)^2 \, 1_{8\times 8} \ , \quad ZW = WZ \ . \tag{5.49}$$

Suppose that $\partial_a\Phi$ is spacelike, then the backgrounds preserve 4 supersymmetries provided that

$$\pm\sqrt{(\partial_a\Phi)^2} = -H_{627} \ . \tag{5.50}$$

For each choice of sign, three of the Killing spinors belong to the eigenspace of W with eigenvalue 1 and one Killing spinor belongs to the eigenspace of W with eigenvalue -3. In addition, they satisfy an appropriate projection with respect to Z. There are no solutions for either $\partial_a\Phi$ null or timelike. All these backgrounds have linear dilaton along the spacelike fibre directions of $\mathbb{R}^{2,1}$.

5.4 Geometry

To solve the KSE (5.14) as in the $SU(3)$ holonomy case, we write $x^M = (y^\alpha, x^\mu)$, where x^μ are coordinates of the base space B and y^α are coordinates of the fibre of M. Then, we have

$$\lambda^a = \lambda^a{}_\alpha dy^\alpha + \lambda^a{}_\mu dx^\mu \ , \quad e^i = e^i{}_\mu dx^\mu \ , \quad \partial_\mu = e^i_\mu\partial_i + \lambda^a_\mu\partial_a \ , \tag{5.51}$$

as in the holonomy $SU(3)$ case. Both the spacetime metric g and H are independent from the fibre coordinates y. Next as in [22] set

$$d\tilde{s}^2 = h \, d\mathring{s}^2 \ , \quad \tilde{H} = -\mathring{\star}dh \ , \tag{5.52}$$

where all tensors, including the function h, depend only on the coordinates x^μ, and the Hodge star operation is taken with respect to the metric $d\mathring{s}^2$. To find the geometry of the base space B, one has to determine $d\mathring{s}^2$.

Substituting the above data (5.52) into (5.13), we get

$$h^{-1}\partial_i h = 2\partial_i \Phi \ . \tag{5.53}$$

Thus

$$\partial_\mu \log h = 2(\partial_\mu \Phi - \lambda^a_\mu \partial_a \Phi) = 0 \ . \tag{5.54}$$

Assuming that $\partial_a \Phi \neq 0$, one finds as an integrability condition that

$$\mathcal{F}^a_{\mu\nu}\, \partial_a \Phi = 0 \ , \tag{5.55}$$

ie the curvature of the principal bundle connection must vanish along the direction $\partial_a \Phi$ in $\mathfrak{Lie}\, G$. Integrating locally (5.54), we get

$$2\Phi = \log\, h + Y(y) \ , \tag{5.56}$$

where Y depends only on the fibre coordinates. Since $\partial_a \Phi$ is constant, Φ is at most linear in the y coordinates.

To find the geometry of the base space, note that

$$\hat{\Omega}_\mu{}^i{}_j = e^k{}_\mu \hat{\Omega}_{k,}{}^i{}_j + \lambda^a{}_\mu \hat{\Omega}_{a,}{}^i{}_j = \mathring{\Omega}_{\mu,}{}^i{}_j + \lambda^a{}_\mu H_{aj}{}^i \ , \tag{5.57}$$

where $\mathring{\Omega}$ is the frame connection of the metric $d^2\mathring{s}$ given in (5.52). Since $\hat{\nabla}\omega_r = 0$, the covariant derivative of ω^r along B with respect to the Levi-Civita connection $\mathring{\nabla}$ of the metric $d\mathring{s}^2$ is

$$\mathring{\nabla}_\mu (\omega_r)_{ij} - 2\lambda^a_\mu f_{as} \epsilon^s{}_r{}^t (\omega_t)_{ij} = 0 \ . \tag{5.58}$$

Note that if $\lambda^a_\mu f_{as} \neq 0$, then $\mathcal{L}_a \omega_r \neq 0$. So the 2-forms ω_r on M do not descent as 2-forms on the base space B. Instead, they are 2-forms on B with values on a bundle with connection λ.

It is known that if there is no restriction on the connection λ, all 4-manifolds satisfy (5.58). However in our case, there is a restriction because the self-dual part of the curvature of λ satisfies

$$\mathcal{F}^a + \star\mathcal{F}^a = 2 f^a_r \omega^r \tag{5.59}$$

and f is a constant matrix.

Clearly if $f = 0$, the base space B is hyper-Kähler and the connection λ an anti-self-dual instanton on B with gauge group $\mathfrak{Lie}(G)$. On the other hand if the anti-self dual part of $\mathcal{F}$ vanishes, then B is self-dual Weyl Einstein. This is the Quaternionic Kähler condition in 4 dimensions, see eg [23].

Returning to the general case, the integrability condition of (5.58) is

$$-\mathring{R}_{\mu\nu,}{}^\lambda{}_\rho \mathring{\omega}^r_{\lambda\sigma} + \mathring{R}_{\mu\nu,}{}^\lambda{}_\sigma \mathring{\omega}^r_{\lambda\rho} = 2\mathcal{F}^a_{\mu\nu} f_{as} \epsilon^{sr}{}_t \mathring{\omega}^t_{\rho\sigma} \ . \tag{5.60}$$

This condition implies that the self-dual part of the Weyl tensor vanishes thus B is a anti-self-dual Weyl manifold. In addition, one finds that

$$\begin{aligned} \mathring{R}_{\mu\nu,}{}^{\rho\sigma}\mathring{\omega}^r_{\rho\sigma} &= 4\mathcal{F}^r_{\mu\nu} \ , \\ \mathring{R}_{\mu\nu} &= -2\sum_r \mathcal{F}^a_{\mu\rho} f_{ar}\mathring{\omega}^r_{\sigma\nu}\gamma^{\rho\sigma} \ , \end{aligned} \tag{5.61}$$

where $d\mathring{s}^2 = \gamma_{\mu\nu}dx^\mu dx^\nu$. Observe that if the anti-self-dual part of $\mathcal{F}$ vanishes, then B is Einstein with cosmological constant proportional to f^2.

So far we investigated the general case. Now we shall adapt the above results to the cases (5.8) and (5.10).

5.4.1 $f = wv$

<u>$N = 4$</u>

To begin define $\omega := v_r\omega^r$. Then using (5.58) observe that ω satisfies

$$\mathring{\nabla}\mathring{\omega} = 0 \ . \tag{5.62}$$

Thus B is Kähler. Moreover, the remaining two equations in (5.58) become

$$\mathring{\nabla}_\mu\mathring{\omega}^r_{ij} - 2\lambda_\mu v_s\epsilon^{sr}{}_t\mathring{\omega}^t_{ij} = 0 \ , \quad \lambda := w_a\lambda^a \ . \tag{5.63}$$

These two conditions are automatically satisfied for all the Kähler manifolds, in fact the other two ω's are sections of $K \otimes L$, where K the canonical bundle on B and L is the line bundle with connection λ. The only additional condition is the analogue of (5.59) which implies that

$$\mathcal{F} + \star\mathcal{F} = 2w^2\omega \ , \tag{5.64}$$

where $\mathcal{F}$ is the curvature of λ. In general B is not Einstein. But if in addition the anti-self-dual part of $\mathcal{F}$ vanishes, then B is Kähler-Einstein with non-vanishing cosmological constant, and the Ricci tensor is given by

$$\mathring{R}_{\mu\nu} = 2w^2\gamma_{\mu\nu} \ . \tag{5.65}$$

The sign of the curvature of B depends on whether w is spacelike or timelike. If w is null, then B is hyper-Kähler. The only negative curved case occurs whenever $\mathfrak{Lie}\, G = \mathbb{R}^{2,1} \oplus \mathfrak{su}(2)$.

Thus, if $f = wv$, B is *Kähler*. In particular $\mathrm{hol}(\mathring{\nabla}) \subseteq U(2)$. Moreover the curvature of the connection along the central element of $\mathfrak{Lie}\, G$ spanned by w, $\lambda = w_a\lambda^a$, satisfies the *Hermitian Einstein condition with cosmological constant*, ie the self-dual part is given by (5.64). The remaining components of $\mathcal{F}^a$ are *anti-self-dual.*

There is a special case of solutions with 4 supersymmetries and B *hyper-Kähler.* These arise for example whenever $w = 0$. Moreover in this case $\mathcal{F}$ is *anti-self-dual.* Such solutions exist for all Lie algebras in (5.1) unless their parameters are restricted such

that the solutions admit 8 supersymmetries. Typically the dilaton is linear along the fibre directions of spacetime but there are also $N = 4$ solutions with constant dilaton.

$\underline{N = 8}$

The $N = 8$ solutions have been classified in [27]. B is *hyper-Kähler* and $\mathcal{F}$ is anti-self-dual with gauge group Lie algebra given in (5.1). The dilaton is constant along the fibre directions of spacetime.

$\mathfrak{Lie}\, G/N$	1	2	3	4	5	6	7	8
$\mathbb{R}^{2,1} \oplus \mathfrak{su}(2)$	–	–	–	HK, K, ASW	–	–	–	–
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^3$	–	–	–	K	–	–	–	–
$\mathfrak{cw}_4 \oplus \mathbb{R}^2$	–	–	–	HK, K	–		–	–
$\mathfrak{cw}_6$	–	–	–	HK, K	–	–	–	HK
$\mathbb{R}^{5,1}$	–	–	–	HK, K	–		–	HK
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$	–	ASW	–	–	–	ASW	–	HK

Table 3: N is the number of supersymmetries. ASW (Anti-self-dual Weyl), K (Kähler) and HK (hyper-Kähler) refers to the geometry of the 4-dimensional base space B of the associated supersymmetric background. The entries – do not occur.

5.4.2 f given in (5.10)

$\underline{N = 2, 4, 6}$

The analysis is identical to that given in the beginning of the section for the general case. The only difference is that f is now restricted to be non-vanishing only on the $\mathfrak{su}(2)$ subalgebra of $\mathbb{R}^{2,1} \oplus \mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$. As we have already remarked the self-dual part of the Weyl tensor of B vanishes. In general, $\mathrm{hol}(\mathring{\nabla}) \subseteq Spin(4)$. The components $\mathcal{F}^r$ of $\mathcal{F}^a$ along the $\mathfrak{su}(2)$ subalgebra of $\mathbb{R}^{2,1} \oplus \mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$ satisfy a Hermitian-Einstein type of condition (5.59). The remaining anti-self-dual components of $\mathcal{F}^a$ are not restricted.

6 Conclusions

We have simplified the solution of the KSEs of heterotic backgrounds for which the connection with skew-torsion $\hat{\nabla}$ has holonomy contained in a compact group. In particular, we have shown that if $dH = 0$ and the field equations are satisfied, then there are restrictions on the fractions of supersymmetry that can occur and these depend on the Lie algebra of isometries that acts on the backgrounds. We have collected an outline of our results in table 4. This should be compared with table 2 of [3] which has been composed without making any additional assumptions. The absence of $N = 3$ solutions in the $SU(3)$ case and $N = 7$ solutions in the $SU(2)$ case is a local result and reminiscent to those in [19, 20] for 10- and 11-dimensional supergravities. We have not ruled out the possibility that such backgrounds could exist after a discrete identification of solutions

with 4 and 8 supersymmetries, respectively, see [21]. Observe in addition that there are no solutions in the $SU(2)$ case with 3 and 5 supersymmetries.

$\mathrm{hol}(\hat{\nabla})$	N
$Spin(7) \ltimes \mathbb{R}^8$	1
$SU(4) \ltimes \mathbb{R}^8$	$\nearrow$, 2
$Sp(2) \ltimes \mathbb{R}^8$	$\nearrow$, $\nearrow$, 3
$\times^2 SU(2) \ltimes \mathbb{R}^8$	$\nearrow$, $\nearrow$, $\nearrow$, 4
$SU(2) \ltimes \mathbb{R}^8$	$\nearrow$, $\nearrow$, $\nearrow$, $\nearrow$, 5
$U(1) \ltimes \mathbb{R}^8$	$\nearrow$, $\nearrow$, $\nearrow$, $\nearrow$, $\nearrow$, 6
$\mathbb{R}^8$	$\nearrow$, $\nearrow$, $\nearrow$, $\nearrow$, $\nearrow$, $\nearrow$, $-$, 8
G_2	1, 2
$SU(3)$	1, 2, $-$, 4
$SU(2)$	$-$, 2, $-$, 4, $-$, 6, $-$, 8
$\{1\}$	8, 10, 12, 14, 16

Table 4: In the columns are the holonomy groups that arise from the solution of the gravitino KSE and the number N of supersymmetries, respectively. $\nearrow$ and $-$ denote the entries in table 2 of [3] that are special cases of backgrounds for which all parallel spinors are Killing and those that do not occur, respectively.

We have also examined the geometry of the solutions. In all these cases, the spacetime is a principal bundle with fibre group G and base space B. We have determined all the groups G that occur and identify the geometry of B in all cases. In the G_2 case, B admits a G_2 structure compatible with a connection with skew-symmetric torsion. In the $SU(3)$ case, B admits either a $SU(3)$ or a $U(3)$ structure again compatible with a connection with skew-symmetric torsion. B is either a complex or an almost complex manifold. In the $SU(2)$ case, B can either be Kähler, or hyper-Kähler or anti-self-dual Weyl manifold. The latter condition includes the 4-dimensional quaternionic Kähler manifolds which in addition are required to be Einstein. We have also found the conditions on the curvature $\mathcal{F}$ of the connection that twists G over B imposed by supersymmetry. These are typically instanton-like conditions in dimension 7, 6 and 4. Furthermore, we examined the properties of the dilaton for these backgrounds. In particular, we found that some of them exhibit a linear dilaton along the fibre G of spacetime.

There is a worldvolume interpretation for all these solutions. In particular, they can be thought as gauged WZW models with group manifold G over B. The action that it is gauged is the left action which is anomalous. However since the gauge fields are composite, this anomaly can be canceled from a contribution form the base space. The details are similar to those in [27] as applied to backgrounds that preserve 8 supersymmetries.

The supersymmetric heterotic backgrounds exhibit a rich geometric structure. Many explicit solutions are known, see eg [22, 24, 25, 26]. There is also a classification of all solutions with 8 supersymmetries [27] and a large class of solutions with 4 supersymmetries can be understood [28]. Further progress can be made to construct new examples. The main question is to find base manifolds B of the spacetime that have the prescribed

properties required by supersymmetry. In most cases, there are no general methods for the construction of such manifolds. The development of such methods is a problem for the future.

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Appendix A Dilatino KSE on group manifolds

It is instructive solve the dilatino KSE in the case that depends only on the dilaton and the Lie structure constants H_{abc}. This has already been done for the group manifold models in [6]. Here we shall adapt this to the $\mathrm{hol}(\hat\nabla) \subseteq SU(2)$ case. The holonomy $\mathrm{hol}(\hat\nabla) \subseteq SU(3)$ is straightforward. If $f = 0$ in the $\mathrm{hol}(\hat\nabla) \subseteq SU(2)$ case, then $\mathcal{F}$ is anti-self-dual. The dilatino Killing spinor equation becomes

$$(\Gamma^a \partial_a \Phi - \frac{1}{12} H_{abc} \Gamma^{abc})\epsilon = 0 \ . \tag{A.1}$$

There are three cases to consider depending on whether $\partial_a\Phi$ is (i) $(\partial_a\Phi)^2 \neq 0$, ie space- or time-like[9] , (ii) $(\partial_a\Phi)^2 = 0$ but $\partial_a\Phi \neq 0$, ie null, or (iii) $\partial_a\Phi = 0$.

$\underline{(\partial_a\Phi)^2 \neq 0}$

Acting with $\Gamma^a\partial_a\Phi$ on (A.1) and using (5.5), one finds that

$$\big[(\partial_a\Phi)^2 - \frac{1}{12}\partial_{a_1}\Phi\, H_{a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}\big]\epsilon = 0 \ . \tag{A.2}$$

Next set

$$A = \frac{1}{12}\partial_{a_1}\Phi\, H_{a_2a_3a_4}\Gamma^{a_1a_2a_3a_4} \tag{A.3}$$

and using that H_{abc} are the structure constants of a metric Lie algebra observe that

$$A^2 = \frac{1}{24}(\partial_a\Phi)^2 H_{abc}H^{abc} 1_{8\times 8} \tag{A.4}$$

For A to have real non-vanishing eigenvalues

$$(\partial_a\Phi)^2 H_{abc}H^{abc} > 0 \tag{A.5}$$

so $\partial_a\Phi$ and H_{abc} are either both spacelike or both time-like.

There are no solutions if H is timelike. To see this observe that H_{abc} is timelike only when $\mathfrak{Lie}(G) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h}$ in (5.1) for $\mathfrak{h} = \mathfrak{su}(2)$ or $\mathbb{R}^3$. For $\mathfrak{h} = \mathfrak{su}(2)$, (A.5) is incompatible with (5.5). Moreover for $\mathfrak{h} = \mathbb{R}^3$, (5.5) implies that $\partial_a\Phi$ is spacelike and so (A.5) is not satisfied.

[9] From now on we shall use the notation $(\partial_a\Phi)^2 = \eta^{ab}\partial_a\Phi\partial_b\Phi$ and $|\partial_a\Phi| = \sqrt{|(\partial_a\Phi)^2|}$.

It remains to see whether there are solutions if H is space-like. If H is space-like (A.5) is satisfied provided that $\partial_a\Phi$ is space-like as well. In such case there are solutions preserving 4 supersymmetries provided

$$(\partial_a\Phi)^2 = \frac{1}{24}H_{abc}H^{abc} > 0. \tag{A.6}$$

Moreover (A.1) is equivalent to the projector

$$\big(1 - \frac{\partial_{a_1}\Phi\, H_{a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}}{12(\partial_a\Phi)^2}\big)\epsilon = 0 \ . \tag{A.7}$$

Next observe that $A^2 = 0$, $A \neq 0$, iff H_{abc} is nilpotent. It is easy to see from this that consistency requires that $\partial_a\Phi$ must also be nilpotent. Thus such backgrounds are included in the solutions which shall investigate below. Therefore, the only solutions that can be arranged to satisfied the conditions (A.5) and (A.6), and so admit 4 supersymmetries, have isometry algebra

$$\mathbb{R}^{3,1} \oplus \mathfrak{su}(2) \ . \tag{A.8}$$

$\underline{\partial_a\Phi \text{ null}}$

If $\partial_a\Phi$ is null, then $A\epsilon = 0$. Now acting on (A.1) with $H_{abc}\Gamma^{abc}$ and using $A\epsilon = 0$, one finds that

$$H_{abc}H^{abc}\epsilon = 0 \tag{A.9}$$

and so for solutions to exist H_{abc} must also be null.

Without loss of generality the only non-vanishing component of $\partial_a\Phi$ can be taken to be $\partial_+\Phi$. In such case (5.5) implies that the non-vanishing components of H_{abc} are $H_{+a'b'}$. In such case, the dilatino KSE is equivalent to the light-cone projection $\Gamma^+\epsilon = 0$. The only groups listed in (5.1) that admit null structure constants are

$$\mathbb{R}^{5,1}\ , \quad \mathfrak{cw}_4 \oplus \mathbb{R}^2\ , \quad \mathfrak{cw}_6\ . \tag{A.10}$$

As in the previous case, these backgrounds preserve strictly 4 supersymmetries.

$\underline{\partial_a\Phi = 0}$

If the dilaton is constant, it is easy to see that (A.1) has solutions provided H_{abc} is null. The Lie algebras

$$\mathbb{R}^{5,1}\ , \quad \mathfrak{cw}_4 \oplus \mathbb{R}^2\ , \quad \mathfrak{cw}_6\ , \tag{A.11}$$

in (5.1) have null structure constants. It turns out that the backgrounds with isometry algebras

$$\mathfrak{cw}_4 \oplus \mathbb{R}^2\ , \quad \mathfrak{cw}_6\ , \tag{A.12}$$

preserve strictly 4 supersymmetries provided that the structure constants of $\mathfrak{cw}_6$ are *not* self-dual, while

$$\mathbb{R}^{5,1}\ , \quad \mathfrak{cw}_6\ , \tag{A.13}$$

preserve all 8 supersymmetries provided that the structure constants of $\mathfrak{cw}_6$ are self-dual.

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